Exploring the Stratified Shortest-Paths Problem

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This Talk

Motivation

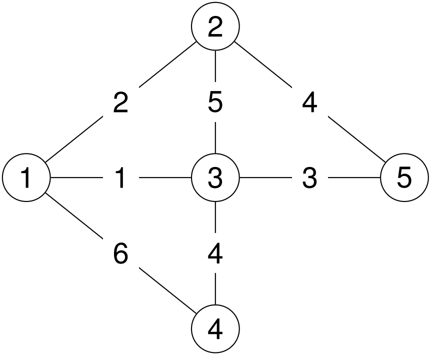
- There is a long history of algebraic approaches to solving path problems in graphs.
- Question : Can BGP be cast in a way that falls within this tradition?

Sources

[Gri10] The Stratified Shortest-Paths Problem
COMSNETS (January, 2010)
TGG

[SG10] Routing in Equilibrium
Math. Theory of Networks and Systems (July, 2010)
João Luís Sobrinho and TGG

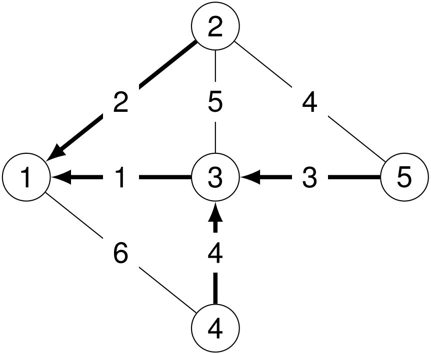
Shortest paths example, $\text{sp} = (\mathbb{N}^{\infty},\ \text{min},\ +)$



The adjacency matrix

$$\mathbf{A} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{ccccc} \infty & 2 & 1 & 6 & \infty \\ 2 & \infty & 5 & \infty & 4 \\ 1 & 5 & \infty & 4 & 3 \\ 6 & \infty & 4 & \infty & \infty \\ \infty & 4 & 3 & \infty & \infty \end{array}\right] \end{array}$$

Shortest paths example, continued



Bold arrows indicate the shortest-path tree rooted at 1.

The routing matrix

$$\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 0 & 3 & 7 & 4 \\ 1 & 3 & 0 & 4 & 3 \\ 5 & 7 & 4 & 0 & 7 \\ 4 & 4 & 3 & 7 & 0 \end{bmatrix} \end{array}$$

Matrix **R** solves this global optimality problem:

$$\mathbf{R}(i,\ j) = \min_{p \in P(i,\ j)} w(p),$$

where $P(i,\ j)$ is the set of all paths from i to j.

Semirings

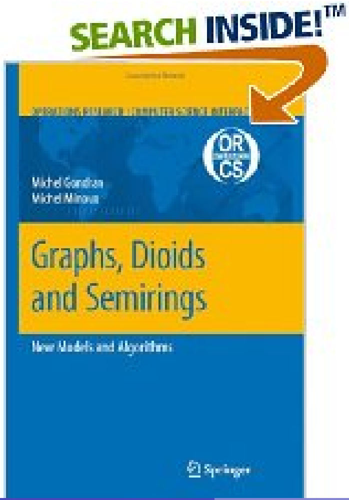
A few examples

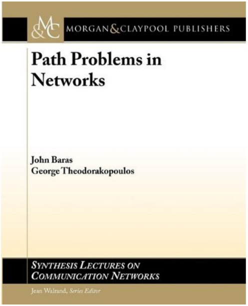
name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$	possible routing use
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0	minimum-weight routing
bw	$\mathbb{N}^{\infty}$	max	min	0	∞	greatest-capacity routing
rel	$[0, 1]$	max	$\times$	0	1	most-reliable routing
use	$\{0, 1\}$	max	min	0	1	usable-path routing
	2^W	$\cup$	$\cap$	$\{\}$	W	shared link attributes?
	2^W	$\cap$	$\cup$	W	$\{\}$	shared path attributes?

Path problems focus on global optimality

$$\mathbf{A}^*(i, j) = \bigoplus_{p \in P(i, j)} w(p)$$

Recomended Reading





What algebraic properties are associated with global optimality?

Distributivity

$$\begin{array}{lcrcl} \mathrm{L.D} & : & a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c), \\ \mathrm{R.D} & : & (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c). \end{array}$$

What is this in $\mathrm{sp} = (\mathbb{N}^{\infty},\ \min,\ +)$?

$$\begin{array}{lcrcl} \mathrm{L.DIST} & : & a + (b \min c) & = & (a + b) \min (a + c), \\ \mathrm{R.DIST} & : & (a \min b) + c & = & (a + c) \min (b + c). \end{array}$$

(Left) Local Optimality

Say that $\mathbf{L}$ is a left-locally optimal solution when

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{L}(i,\ j) = \bigoplus_{q \in V} \mathbf{A}(i,\ q) \otimes \mathbf{L}(q,\ j) = \bigoplus_{(i,\ q) \in E} w(i,\ q) \otimes \mathbf{L}(q,\ j),$$

In other words, $\mathbf{L}(i,\ j)$ is the best possible value given the values $\mathbf{L}(q,\ j)$, for all out-neighbors q of source i.

(Right) Local Optimality

Say that $\mathbf{R}$ is a left-locally optimal solution when

$$\mathbf{R} = (\mathbf{R} \otimes \mathbf{A}) \oplus \mathbf{I}.$$

That is, for $i \neq j$ we have

$$\mathbf{R}(i,\ j) = \bigoplus_{q \in V} \mathbf{R}(i,\ q) \otimes \mathbf{A}(q,\ j) = \bigoplus_{(q,\ j) \in E} \mathbf{R}(i,\ q) \otimes w(q,\ j),$$

In other words, $\mathbf{R}(i,\ j)$ is the best possible value given the values $\mathbf{R}(q,\ j)$, for all in-neighbors q of destination j.

With and Without Distributivity

With

For (well behaved) Semirings, the three optimality problems are essentially the same — locally optimal solutions are globally optimal solutions.

$$\mathbf{A}^* = \mathbf{L} = \mathbf{R}$$

Without

Suppose that we drop distributivity and $\mathbf{A}^*$, $\mathbf{L}$, $\mathbf{R}$ exist. It may be the case they they are all distinct.

A World Without Distributivity

Global Optimality

This has been studied, for example [LT91b, LT91a] in the context of circuit layout. See Chapter 5 of [BT10]. This approach does not play well with (loop-free) hop-by-hop forwarding (need tunnels!)

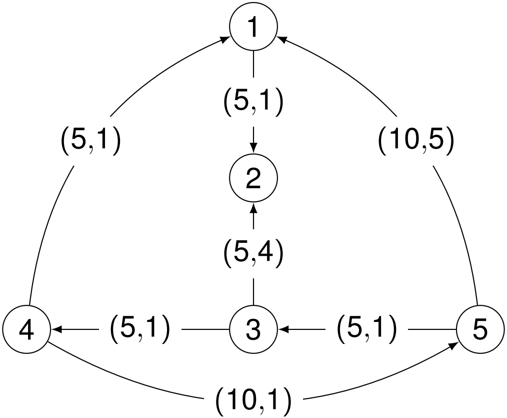
Left Local Optimality

At a very high level, this is the type of problem that BGP attempts to solve!!

Right Local Optimality

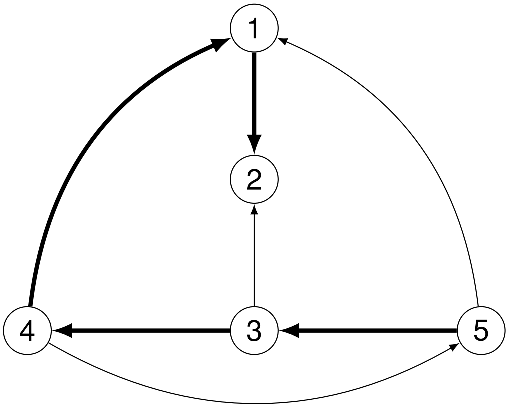
This approach does not play well with (loop-free) hop-by-hop forwarding (need tunnels!)

Example

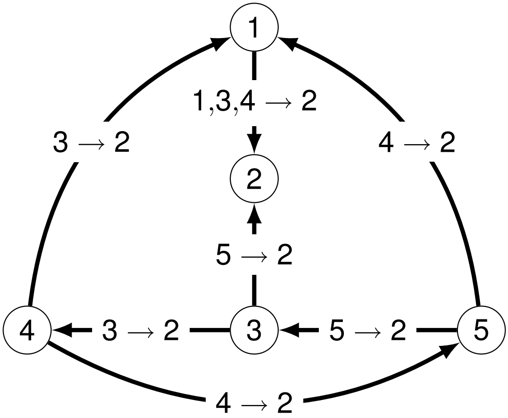


(bandwidth, distance) with lexicographic order (bandwidth first).

Left-locally optimal paths to node 2



Right-locally optimal paths to node 2



Functions on arcs

From $(S, \oplus, \otimes, \overline{0}, \overline{1})$ to $(S, \oplus, F, \overline{0}, \overline{1})$

- Replace $\otimes$ with $F \subseteq S \rightarrow S$,
- Replace

$$\mathrm{L.D} \quad : \quad a \otimes (b \oplus c) \quad = \quad (a \otimes b) \oplus (a \otimes c)$$

  with

$$\mathrm{D} \quad : \quad f(b \oplus c) \quad = \quad f(b) \oplus f(c)$$

- Path weight is now

$$\begin{aligned} w(p) &= g_{(v_0,\ v_1)}(g_{(v_1,\ v_2)} \cdots (g_{(v_{k-1},\ v_k)}(\overline{1}) \cdots)) \\ &= (g_{(v_0,\ v_1)} \circ g_{(v_1,\ v_2)} \circ \cdots \circ g_{(v_{k-1},\ v_k)})(\overline{1}) \end{aligned}$$

What accounts for loss of distributivity?

- Algebras can be constructed from component algebras, and we must be careful. EIGRP is an example [GS03].
- Link weights may be a function of path weight. From

$$w(v_0, v_1, \cdots, v_k) = w(v_0,\ v_1) \otimes w(v_1, \cdots, v_k)$$

to

$$w(v_0, v_1, \cdots, v_k) = g_{(v_0,\ v_1)}(w(v_1, \cdots, v_k)) \otimes w(v_1, \cdots, v_k).$$

This makes distributivity harder to maintain (especially given the kinds of g's natural in a routing context).

What are the conditions needed to guarantee existence of local optima?

For a non-distributed structure $S = (S,\ \oplus,\ F,\ \overline{0},\ \overline{1})$, can be used to find local optima when the following property holds.

Strictly Inflationary

$$\text{S.INFL} : \forall a,\ b \in S : a \neq \overline{0} \implies a < b \otimes a$$

where $a \leq b$ means $a = a \oplus b$.

Useful properties

$$(S,\ \oplus,\ F,\ \overline{0},\ \overline{1})$$

property	definition
D	$\forall a, b \in S,\ f \in F\ :\ f(a \oplus b) = f(a) \oplus f(b)$
INFL	$\forall a \in S,\ f \in F\ :\ a \leq f(a)$
S.INFL	$\forall a \in S,\ F \in F\ :\ a \neq \overline{0} \implies a < f(a)$
$K_{\overline{0}}$	$\forall a, b \in S,\ f \in F\ :\ f(a) = f(b) \implies (a = b \vee f(a) = \overline{0})$
$C_{\overline{0}}$	$\forall a, b \in S,\ f \in F\ :\ f(a) \neq f(b) \implies (f(a) = \overline{0} \vee f(b) = \overline{0})$

Stratified Shortest-Paths Metrics

Metrics

$$(s,\ d)\ \text{ or }\ \infty$$

- $s \neq \infty$ is a *stratum level* in $\{0, 1,\ 2, \ldots,\ m-1\}$,
- d is a "shortest-paths" distance,
- Routing metrics are compared lexicographically

$$(s_1,\ d_1) < (s_2,\ d_2) \iff (s_1 < s_2) \vee (s_1 = s_2 \wedge d_1 < d_2)$$

Stratified Shortest-Paths Policies

Policy has form $(f,\ d)$

$$
\begin{aligned}
(f,\ d)(s,\ d') &= \langle f(s),\ d + d' \rangle \\
(f,\ d)(\infty) &= \infty
\end{aligned}
$$

where

$$
\langle s,\ t \rangle = \begin{cases} \infty & (\text{if } s = \infty) \\ (s,\ t) & (\text{otherwise}) \end{cases}
$$

Constraint on Policies

$$(f,\ d)$$

- Either f is inflationary and $0 < d$,
- or f is strictly inflationary and $0 \leq d$.

Why?

$$(\text{S.INFL}(S) \vee (\text{INFL}(S) \wedge \text{S.INFL}(T))) \implies \text{S.INFL}(S \vec{\times}_{\overline{0}} T).$$

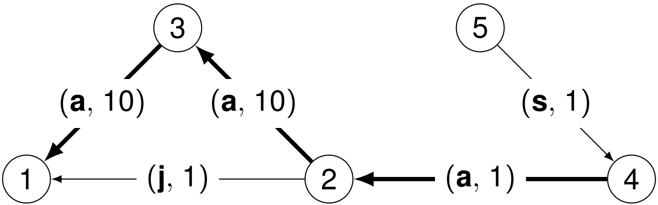
All Inflationary Policy Functions for Three Strata

	0	1	2	D	K_∞	C_∞		0	1	2	D	K_∞	C_∞
a	0	1	2	$\star$	$\star$		**m**	2	1	2			
b	0	1	∞	$\star$	$\star$		**n**	2	1	∞		$\star$	
c	0	2	2	$\star$			**o**	2	2	2	$\star$		$\star$
d	0	2	∞	$\star$	$\star$		**p**	2	2	∞	$\star$		$\star$
e	0	∞	2		$\star$		**q**	2	∞	2			$\star$
f	0	∞	∞	$\star$	$\star$	$\star$	**r**	2	∞	∞	$\star$	$\star$	$\star$
g	1	1	2	$\star$			**s**	∞	1	2		$\star$	
h	1	1	∞	$\star$		$\star$	**t**	∞	1	∞		$\star$	$\star$
i	1	2	2	$\star$			**u**	∞	2	2			$\star$
j	1	2	∞	$\star$	$\star$		**v**	∞	2	∞		$\star$	$\star$
k	1	∞	2		$\star$		**w**	∞	∞	2		$\star$	$\star$
l	1	∞	∞	$\star$	$\star$	$\star$	**x**	∞	∞	∞	$\star$	$\star$	$\star$

Almost shortest paths

	0	1	2	D	K_∞	interpretation
a	0	1	2	$\star$	$\star$	$+0$
j	1	2	∞	$\star$	$\star$	$+1$
r	2	∞	∞	$\star$	$\star$	$+2$
x	∞	∞	∞	$\star$	$\star$	$+3$
b	0	1	∞	$\star$	$\star$	filter 2
e	0	∞	2		$\star$	filter 1
f	0	∞	∞	$\star$	$\star$	filter 1, 2
s	∞	1	2		$\star$	filter 0
t	∞	1	∞		$\star$	filter 0, 2
w	∞	∞	2		$\star$	filter 0, 1

Shortest paths with filters, over $\mathbf{INF}_3$



Note that the path $5, 4, 2, 1$ with weight $(1,\ 3)$ would be the globally best path from node 5 to node 1. But in this case, poor node 5 is left with no path! The locally optimal solution has $\mathbf{R}(5,\ 1) = \infty$.

Both D and $K_{\overline{0}}$

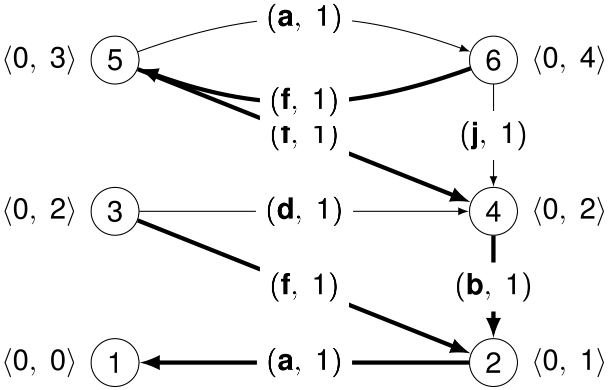
This makes combined algebra distributive!

	0	1	2
a	0	1	2
b	0	1	∞
d	0	2	∞
f	0	∞	∞
j	1	2	∞
l	1	∞	∞
r	2	∞	∞
x	∞	∞	∞

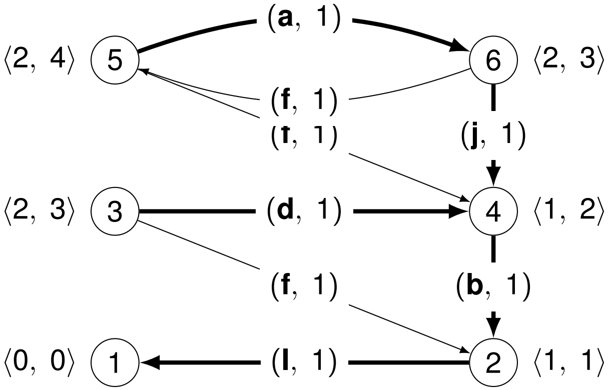
Why?

$$(\mathrm{D}(S) \wedge \mathrm{D}(T) \wedge \mathrm{K}_{\overline{0}}(S)) \implies \mathrm{D}(S \vec{\times}_{\overline{0}} T)$$

Example 1



Example 2



BGP : standard view

- 0 is the type of a *downstream* route,
- 1 is the type of a *peer* route, and
- 2 is the type of an *upstream* route.

	0	1	2
f	0	∞	∞
l	1	∞	∞
o	2	2	2

“Autonomous” policies

	0	1	2	D	K_∞
f	0	∞	∞	$\star$	$\star$
h	1	1	∞	$\star$	
l	1	∞	∞	$\star$	$\star$
o	2	2	2	$\star$	
p	2	2	∞	$\star$	
q	2	∞	2		
r	2	∞	∞	$\star$	$\star$
t	∞	1	∞		$\star$
u	∞	2	2		
v	∞	2	∞		$\star$
w	∞	∞	2		$\star$
x	∞	∞	∞	$\star$	$\star$

Putting BGP in context, summary

Two main differences over previous work on algebraic path problems in graphs.

- Natural to think that link weights are not fixed but are instead a function of the path (route) itself.
  - Very difficult to perserve distributivity with “dependent” link weights.
- When distributivity fails, look for local optimal solutions.
  - This required some new theory.

Open Problems

- Complexity of solving for left-local solutions?
  - Recent result by Sobrinho and Griffin [SG10] : $O(V^3)$ with a greedy algorithm.
  - We know that "path vectoring" will find a solution, but still no known bounds.
- How could the $> \ m!$ policies be expressed/implemented in BGP? Can this be done without giving up some autonomy?
- Other applications of local optimality.

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To appear.