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SIMILARITY AND DIFFEOMORPHISM CLASSIFICATION OF $\mathbf{S^2} \times \mathbf{R}$ MANIFOLDS

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1. Introduction

The 3-space $\mathbf{S^2} \times \mathbf{R}$ is the direct product of the 2-sphere and the real line. The similarity group

$$Sim(\mathbf{S^2} \times \mathbf{R}) := Isom(\mathbf{S^2}) \times Sim(\mathbf{R}) := \{A\} \times \{(a, b)\} \tag{1.1}$$

where $A \in \mathbf{O^3}$ the 3-dimensional orthogonal group acting on $\mathbf{S^2}$; $a \in \mathbf{R} \setminus \{0\}$, $b \in \mathbf{R}$ and $x \mapsto xa + b$ define a similarity of $\mathbf{R}$.
The isometry group

$$Isom(\mathbf{S^2} \times \mathbf{R}) := Isom(\mathbf{S^2}) \times Isom(\mathbf{R}) \tag{1.2}$$

is specified by $a := \pm 1$.

At the similarity classification of $\mathbf{S^2} \times \mathbf{R}$ space groups in [1], the fixed point free isometry groups G, leaving invariant a translation lattice of $\mathbf{R}$, have also been found and listed in infinite series which lead to space forms $\mathbf{S^2} \times \mathbf{R}/G$, i.e. compact manifolds with local $\mathbf{S^2} \times \mathbf{R}$ metric [2],[3],[4] (see our Table 2).

It turns out that - instead of similarity equivariance - the diffeomorphism one

$$G \sim G' = S^{-1}GS \tag{1.3}$$

with a very simple "skew" diffeomorphism S leads to 4 diffeomorphism classes of $\mathbf{S^2} \times \mathbf{R}$ space forms derived first very sketchily in [5]:

2 orientable ones (with fundamental group $\mathbf{Z}$ and $\mathbf{Z_2} \otimes \mathbf{Z_2}$, respectively; here $\otimes$ stands for free product of Coxeter groups)

and 2 nonorientable ones (with $\mathbf{Z_2} \times \mathbf{Z}$ and $\mathbf{Z}$, respectively).

Surprisingly, we find in the book [4] - without any proof - the statement on the existence of one nonorientable manifold, up to diffeomorphism, that admits $\mathbf{S^2} \times \mathbf{R}$ structures. This statement is false then obviously, in the earlier survey [3] we can read the correct numbers.

We are working - in this comparison - on the classification of space forms in the other fibre geometries $\mathbf{H^2} \times \mathbf{R}$, $\widetilde{\mathbf{SL_2R}}$ and $\mathbf{Nil}$ as well.

Although P. Scott [3] has presented a strategy for describing all the Seifert bundles for the four compact $\mathbf{S^2} \times \mathbf{R}$ manifolds, we find it actual to give another

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more complete interpretation which seems to be advantageous for other reason (see also [1] and [2]).

2. $\mathbf{S^2} \times \mathbf{R}$ ISOMETRIES AND SPACE FORMS, BASIC OBSERVATIONS

As we mentioned in the Introduction, an $\mathbf{S^2} \times \mathbf{R}$ *space form* can naturally be defined as a *factor space* $(\mathbf{S^2} \times \mathbf{R})/G$, where G is an *isometry group* of $\mathbf{S^2} \times \mathbf{R}$, containing an *invariant lattice in* $\mathbf{R}$, denoted by L_G, as follows

$$G \triangleright L_G = \langle \tau \rangle,\ \tau : \mathbf{S^2} \times \mathbf{R} \to \mathbf{S^2} \times \mathbf{R},\ (X, x) \mapsto (X, x + t) \tag{2.1}$$

with a minimal $0 < t \in \mathbf{R}$; moreover, G *acts freely* on $\mathbf{S^2} \times \mathbf{R}$ (i.e. without any fixed point) with compact fundamental domain (of non-empty interior).
By a similarity of $\mathbf{S^2} \times \mathbf{R}$ we may assume that $t = 1$. G is called *space form group* or *fundamental group* as well.

$$G := \{A_i \times \kappa_i\} := \{A_i \times (K_i, k_i)\} := \{A_i \times K_i, k_i\} \tag{2.2}$$

where $A_i \in \mathbf{O^3}$ acts on $\mathbf{S^2}$, $\kappa = (K_i, k_i)$ acts on $\mathbf{R}$. Here K_i is either the identity $1_\mathbf{R}$ of $\mathbf{R}$ or the reflection in zero $\overline{1_\mathbf{R}} : x \mapsto -x$. The "linear parts" of G in (2.2) form the *point group*

$$G_0 = \{(A_i \times K_i)\} \tag{2.3}$$

of G. The *translational parts* k_i to $(A_i \times K_i)$ have to satisfy the multiplication formula

$$(A_1 \times K_1, k_1) \circ (A_2 \times K_2, k_2) = (A_1 A_2 \times K_1 K_2, k_1 K_2 + k_2) \tag{2.4}$$

where we have indicated that our transforms act from the right throughout this paper. Formula (2.4) can be derived from the assumed right action, in general:

$$(X, x)(A_i \times \kappa_i) = (XA_i, xK_i + k_i). \tag{2.5}$$

Any isometry of $\mathbf{S^2} \times \mathbf{R}$ is a product of at most 5 reflections. At most 3 reflections (in equator circles of $\mathbf{S^2}$) produce any element of Isom$\mathbf{S^2}$:=Isom$\mathbf{S^2}\times$ Id$\mathbf{R}$, at most 2 reflections(in points of $\mathbf{R}$) are for Isom$\mathbf{R}$:=Id$\mathbf{S^2}\times$ Isom$\mathbf{R}$.
$\mathbf{S^2_i R_j}$ denotes the set of reflections above, where $i = 0 \dots 3,\ j = 0 \dots 2$ (respectively, $i = 0$ and $j = 0$ for Id$(\mathbf{S^2} \times \mathbf{R})$).

Proposition 2.1 *Any space form group G has a finite point group G_0.*
The *proof* is indirect. Since the linear parts of Isom$\mathbf{R}$ contain 2 elements, then $\{A_i\}$ in (2.3) would have infinitely many ones from Isom$\mathbf{S^2}$. But $\mathbf{S^2}$ is compact, and we assumed a lattice $L_G = \langle \tau \rangle \lhd G$. Thus, there does not exist any open set in the compact "shell" $\mathbf{S^2} \times [0, 1]$ (Fig.1) which contains only points not equivalent under the infinitely many transforms $\{\{A_i \times 1_\mathbf{R}, k_i\}, 0 \le k_i < 1\} =: \overline{G}_0 \subset G$. Then G cannot have any fundamental domain with non-empty interior F^0_G, since the infinite disjoint union of $\overline{G}_0$-images of this F^0_G would lie in the compact shell $\mathbf{S^2} \times [0, 2]$, a contradiction.■

Remarks 1, In the proof we did not utilize, that G was fixed point free.
2, If G is not assumed to have a lattice, then it may have infinite point group G_0.

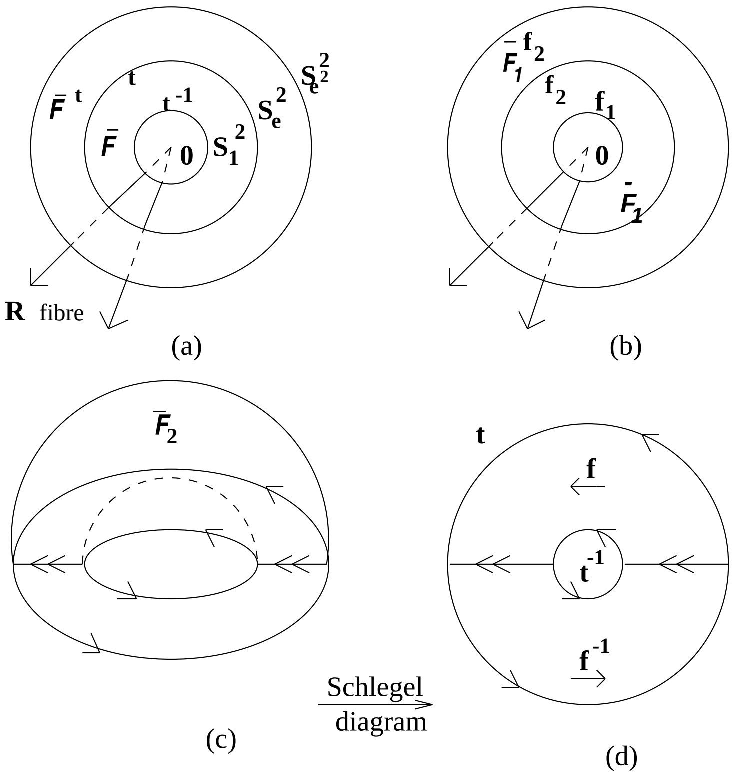

FIGURE 1. $\mathbf{S^2} \times \mathbf{R}$ is modelled in $\mathbf{E}^3_\infty := \mathbf{E^3} \cup \{\infty\}$ where the origin 0 and the infinity ∞ are distinguished. The 0-concentric sphere of Euclidean radius x models the level $\mathbf{S^2} \times \{r\}$ by $r = lnx$. Thus 0 is a joint point $-\infty$ of the $\mathbf{R}$-fibres $\{s\} \times \mathbf{R}$ ($s \in \mathbf{S^2}$) as 0-rays, ∞ is a common point $+\infty \in \{s\} \times \mathbf{R}$. The spherical transforms are usual. The transforms of $\mathbf{R}$ appear as the following "dictionary" translates:

reflection ($\in \mathbf{R_1}$) of $\mathbf{S^2} \times \mathbf{R}$ $\Longleftrightarrow$ sphere inversion of $\mathbf{E}^3_\infty$
in a sphere $\mathbf{S^2} \times \{k\}$ in an 0-centered sphere of radius ϱ
where $k = ln\varrho$

translation ($\in \mathbf{R_2}$) of $\mathbf{S^2} \times \mathbf{R}$ $\Longleftrightarrow$ 0-central similarity of $\mathbf{E}^3_\infty$
with $d \in \mathbf{R}$ from 0 with factor λ
where $d = ln\lambda$

(a) $\bar{\mathcal{F}}$ is a shell describing $\mathbf{Or1}(\mathbf{Z})$, generated by a translation τ pairing the spheres $\mathbf{S}_{t^{-1}}$ and $\mathbf{S}_t$ of $\bar{\mathcal{F}}$ (the letter $\mathbf{S}$ is left in the figure).

(b) $\mathbf{Or2}(\mathbf{Z_2} \otimes \mathbf{Z_2})$ is represented by the shell $\bar{\mathcal{F}}_1$, each of its boundary spheres is paired with itself by an involutive map $\mathbf{f_i} \in \mathbf{S^2_3 R_1}$ ($i = 1, 2$).

(c) Or equivalently, a half shell $\bar{\mathcal{F}}_2$ and its Schlegel diagram in picture (d) describes $\mathbf{Or2}$ by $(\mathbf{f}, \tau - \mathbf{ff}, \mathbf{f}\tau\mathbf{f^{-1}}\tau)$.

With $y \in \mathbf{R}$ and with the usual (geographic) sphere coordinates φ (mod 2π) and $-\frac{\pi}{2} \leq \vartheta \leq \frac{\pi}{2}$, any "screw motion" of $\mathbf{S^2} \times \mathbf{R}$

$$\text{(2.6)} \qquad \mathbf{s} : (\varphi, \vartheta, y) \mapsto (\varphi + \alpha, \vartheta, y + a); \ \frac{\alpha}{2\pi} \in \mathbf{Q}^*; \ 0 < a \in \mathbf{R}$$

generates a cyclic group $G := \langle \mathbf{s} \rangle$ with infinite point group G_0 ($\mathbf{Q}^*$ denotes the set of irrational numbers). The orbit space $\mathbf{S^2} \times \mathbf{R}/\langle \mathbf{s} \rangle$ can be represented by the "shell-like" compact fundamental domain $\overline{\mathcal{F}} = \mathbf{S^2} \times [0, a]$ with a pairing (the bar refers to this) of its 0- and a-level by (2.6). See Fig.1 for an analogous picture.

G is fixed point free, i.e. we get a compact manifold with local $\mathbf{S^2} \times \mathbf{R}$-metric. Then

$$\text{(2.7)} \qquad \mathbf{S^2} \times \mathbf{R}/\langle \mathbf{s} \rangle \sim \overline{\mathcal{F}}$$

may be called an $\mathbf{S^2} \times \mathbf{R}$ space form in general sense. Then we promptly have uncountable many similarity classes of $\mathbf{S^2} \times \mathbf{R}$ space forms, parametrized just by the irrational number $\alpha/2\pi \in (0, 1/2)$. The similarity parameter a in (2.6) is not essential.

As we have promised in the introduction, we can formulate the illustrative
Proposition 2.2 *Any* $\mathbf{S^2} \times \mathbf{R}/\langle \mathbf{s} \rangle$ *above is diffeomorphic to* $\mathbf{S^2} \times \mathbf{R}/\langle \tau \rangle$, *in* (2.1) *with* $t = 1$ *by the "skew" transform*

$$\text{(2.8)} \qquad S: \ \mathbf{S^2} \times \mathbf{R} \to \mathbf{S^2} \times \mathbf{R} : (\overline{\varphi}, \overline{\vartheta}, \overline{y}) \mapsto (\varphi, \vartheta, y) := (\overline{\varphi} + \overline{y}\alpha, \overline{\vartheta}, \overline{y}a)$$

so that $\mathbf{s} = S^{-1}\tau S$.

Proof (see the symbolic Fig.2). By our conventions for the coordinates of $\mathbf{S^2} \times \mathbf{R}$ and for the parameters of $\mathbf{s}$ in (2.6), the skew transform S is a bijection, indeed. For this $\overline{y} \leftrightarrow y$, $\overline{\vartheta} \leftrightarrow \vartheta$ are obvious. If $\overline{\varphi}$ runs over an interval of length 2π, then so does $\varphi = \overline{\varphi} + \overline{y}\alpha$ for any fixed $\overline{y}$. Moreover, the Jacobian

$$\text{(2.8')} \qquad \frac{\partial(\varphi, \vartheta, y)}{\partial(\overline{\varphi}, \overline{\vartheta}, \overline{y})} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$$

is constant.

Since $\tau : (\overline{\varphi}, \overline{\vartheta}, \overline{y}) \mapsto (\overline{\varphi}, \overline{\vartheta}, \overline{y} + 1)$ is a unit translation, thus

$$\text{(2.9)} \qquad (\varphi, \vartheta, y) \xrightarrow{S^{-1}} (\overline{\varphi}, \overline{\vartheta}, \overline{y}) \xrightarrow{\tau} (\overline{\varphi}, \overline{\vartheta}, \overline{y} + 1) \xrightarrow{S} (\overline{\varphi} + (\overline{y} + 1)\alpha, \overline{\vartheta}, (\overline{y} + 1)a) =$$

$$= (\varphi + \alpha, \vartheta, y + a) \ \textit{as at } \mathbf{s}. \blacksquare$$

Remarks 3, As before we can see that $\mathbf{s}$ in (2.6) is similarity equivariant to $\overline{\mathbf{s}} : (\overline{\varphi}, \overline{\vartheta}, \overline{y}) \mapsto (\overline{\varphi} - \alpha, \overline{\vartheta}, \overline{y} + 1)$ by the similarity

$$\text{(2.10)} \qquad \sigma : (\overline{\varphi}, \overline{\vartheta}, \overline{y}) \mapsto (\varphi, \vartheta, y) := (-\overline{\varphi}, \overline{\vartheta}, \overline{y}a);$$

$$\mathbf{s} = \sigma^{-1}\overline{\mathbf{s}}\sigma$$

holds indeed.

Thus we have proven all statements in Rem. 2,.

4, The screw motion, with $2 \le q \in \mathbf{N}$ (for natural numbers)

$$\mathbf{s} : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{2\pi}{q}, \vartheta, y + \frac{k}{q}) \in \mathbf{S^2_2 R_2} \tag{2.11}$$

with greatest common divisor (g.c.d) $(k, q) = 1$, and $1 \le k \le \lfloor \frac{q}{2} \rfloor$ (the lower integer part (l.i.p) of $\frac{q}{2}$) and the lattice $\langle \tau \rangle$ in (2.1) with $t = 1$, determine an orientable space form $\mathbf{S^2} \times \mathbf{R}/G$ in our original (restricted) definition. These lie in different similarity classes for different pairs q, k above. However, they are all diffeomorphic to $\mathbf{S^2} \times \mathbf{R}/\langle \tau \rangle$ by Prop. 2.2, so with the cyclic fundamental group $G \sim \mathbf{Z}$.
To this we consider the transform

$$\mathbf{s}^u \tau^{-v} : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{2\pi u}{q}, \vartheta, y + \frac{ku}{q} - v) \tag{2.12}$$

from G, where $ku - qv = 1$ can be achieved, since $(k, q) = 1$, by appropriate integers u, v with $0 < u < q$ and $0 \le v < k$. Different k_1 and k_2 cannot yield the same u in (2.12), else q would divide u, a contradiction. However, k and $q - k$ lead to equivariant groups by similarity of type (2.10).■
The diffeomorphism class, represented by $\mathbf{S^2} \times \mathbf{R}/\langle \tau \rangle$ by Prop. 2.2 will be denoted by $\mathbf{Or1}(\mathbf{Z})$. We summarize the previous results in
Proposition 2.3 *The diffeomorphism class* $\mathbf{Or1}(\mathbf{Z})$ *of* $\mathbf{S^2} \times \mathbf{R}$ *space forms contains the infinite series of similarity classes described exactly in Rem.4, formula (2.11).*
The *proof* is completed by observing the angular invariant $\alpha = \frac{2\pi u}{q} = -\frac{2\pi(q-u)}{q}$ (mod 2π) belonging to the shortest translation part of length $\frac{1}{q}$ in (2.12).
Moreover, we shall find $\mathbf{Or2}(\mathbf{Z_2} \otimes \mathbf{Z_2})$ as a diffeomorphism class, containing exactly one similarity class of the remaining orientation preserving fixed point free isometry groups of $\mathbf{S^2} \times \mathbf{R}$.■
Or2 will be represented by the group denoted by **7, 1.III.1**(0) in [1]. The fundamental group $G \sim \mathbf{Z_2} \otimes \mathbf{Z_2}$ will be a free product of two Coxeter groups: $G = \langle \mathbf{f_1} \rangle \otimes \langle \mathbf{f_2} \rangle$. Here

$$\mathbf{f_1} : (\varphi, \vartheta, y) \mapsto (\varphi + \pi, -\vartheta, -y) \in \mathbf{S^2_3 R_1} \tag{2.13}$$

$$\mathbf{f_2} : (\varphi, \vartheta, y) \mapsto (\varphi + \pi, -\vartheta, -y + 1) \in \mathbf{S^2_3 R_1}$$

are two involutive generators of G whose elements are

$$\mathbf{1},\ \tau := \mathbf{f_1 f_2},\ \tau^{-1} := \mathbf{f_2 f_1}, \dots, \tau^n, \tau^{-n}, \dots,\ n = 0, 1, \dots\ (\sim \mathbf{Z}) \tag{2.14}$$

$$\tau^k \mathbf{f_1} = \mathbf{f_1} \tau^{-k}, \dots, \tau^{-k} \mathbf{f_2} = \mathbf{f_2} \tau^k, \dots,\ k = 0, 1, \dots$$

By other words: G is an *infinite dihedral group*, or G is a *free Coxeter group of* 2 *generators* (see Fig.1 for 2 geometric presentations of **Or2**).

3. A systematic enumeration of $\mathbf{S^2} \times \mathbf{R}$ space forms

In Table 1 there are listed the finite isometry groups A of $\mathbf{S^2}$ in different notations, from which we prefer the 2-orbifold signatures of Macbeath and Conway, equivalent to each other. Here the factor surface $\mathbf{S^2}/A$ are characterized by the A-orbits of $\mathbf{S^2}$. Any fundamental domain $\overline{\mathcal{F}}_A$ with a side pairing - as usual - provides us a more visual picture (Fig.3).
E.g. the group

$$\mathbf{1q} - (+, 0; [q, q]; \{\}),\ q \geq 1\ -\ \mathbf{q}, \mathbf{q} \tag{3.1}$$

is generated by

$$\mathbf{r} : (\varphi, \vartheta) \mapsto (\varphi + \frac{2\pi}{q}, \vartheta)$$

a q-fold rotation of $\mathbf{S^2}$.
A 2-gon (digon) with $\frac{2\pi}{q}$ angles at the opposite poles and with pairing the (may be bent) sides by $\mathbf{r}$, will topologically be an orientable $(+)$ surface of genus 0 (a sphere), where the two opposite q-fold rotational centres are distinguished (as two cone points) by $\frac{2\pi}{q}$ angular neighbourhood of $\mathbf{S^2}$ at each pole (Fig.3).

	Macbeath signature	H. Weyl	Schoen-flies	Coxeter-Moser	Conway
1q	$(+, 0; [q, q]; \{\})$ $q \geq 1$	C_q	C_q	$[q]^+$	q, q
2q	$(+, 0; [\,]; \{(q, q)\})$ $q \geq 2$	D_qC_q	C_{qv}	[q]	$*q, q$
3q	$(+, 0; [2, 2, q]; \{\})$ $q \geq 2$	D_q	D_q	$[2,q]^+$	$2, 2, q$
4qo	$(+, 0; [\,]; \{(2, 2, q)\})$ $q \geq 3$	$D_{2q}D_q$	D_{qh}	[2,q]	$*2, 2, q$
4qe	$(+, 0; [\,]; \{(2, 2, q)\})$ $q \geq 2$	$D_q \times I$	D_{qh}	[2,q]	$*2, 2, q$
5qo	$(+, 0; [q]; \{(1)\})$ $q \geq 1$	$C_{2q}C_q$	C_{qh}	$[2,q^+]$	$q*$
5qe	$(+, 0; [q]; \{(1)\})$ $q \geq 2$	$C_q \times I$	C_{qh}	$[2,q^+]$	$q*$
6qo	$(+, 0; [2]; \{(q)\})$ $q \geq 3$	$D_q \times I$	D_{qd}	$[2^+,2q]$	$2 * q$
6qe	$(+, 0; [2]; \{(q)\})$ $q \geq 2$	$D_{2q}D_q$	D_{qd}	$[2^+,2q]$	$2 * q$
7qo	$(-, 1; [q]; \{\})$ $q \geq 1$	$C_q \times I$	S_{2q}	$[2^+,2q^+]$	$q\otimes$
7qe	$(-, 1; [q]; \{\})$ $q \geq 2$	$C_{2q}C_q$	S_{2q}	$[2^+,2q^+]$	$q\otimes$
8	$(+, 0; [2, 3, 3]; \{\})$	A_4	T	$[3,3]^+$	$2, 3, 3$
9	$(+, 0; [2, 3, 4]; \{\})$	S_4	O	$[3,4]^+$	$2, 3, 4$
10	$(+, 0; [2, 3, 5]; \{\})$	A_5	I	$[3,5]^+$	$2, 3, 5$
11	$(+, 0; [\,]; \{(2, 3, 3)\})$	S_4A_4	T_d	[3,3]	$*2, 3, 3$
12	$(+, 0; [\,]; \{(2, 3, 4)\})$	$S_4 \times I$	O_h	[3,4]	$*2, 3, 4$
13	$(+, 0; [\,]; \{(2, 3, 5)\})$	$A_5 \times I$	I_h	[3,5]	$*2, 3, 5$
14	$(+, 0; [3]; \{(2)\})$	$A_4 \times I$	T_h	$[3^+,4]$	$3 * 2$

Table 1.

To form appropriate $\mathbf{S^2} \times \mathbf{R}$ space form group G from $\mathbf{q}, \mathbf{q}$ above, we choose first a point group G_0 by (2.3) then the translational parts by (2.4), so that the

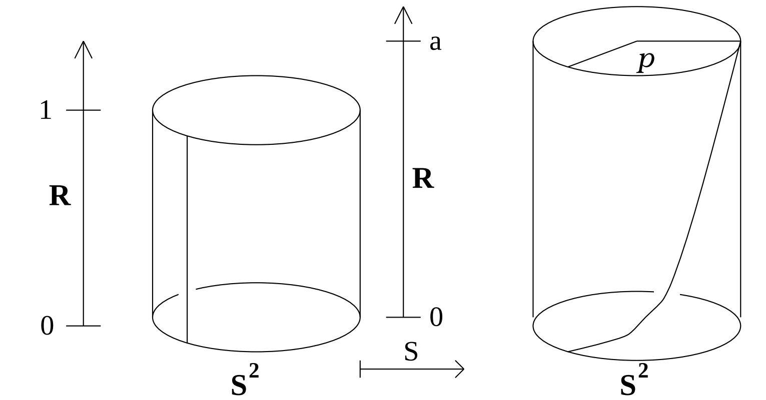

FIGURE 2. Symbolic picture for diffeomorphism equivariance by a skew transform S. $\mathbf{s} = S^{-1}\tau S$. Here p denotes the angle α

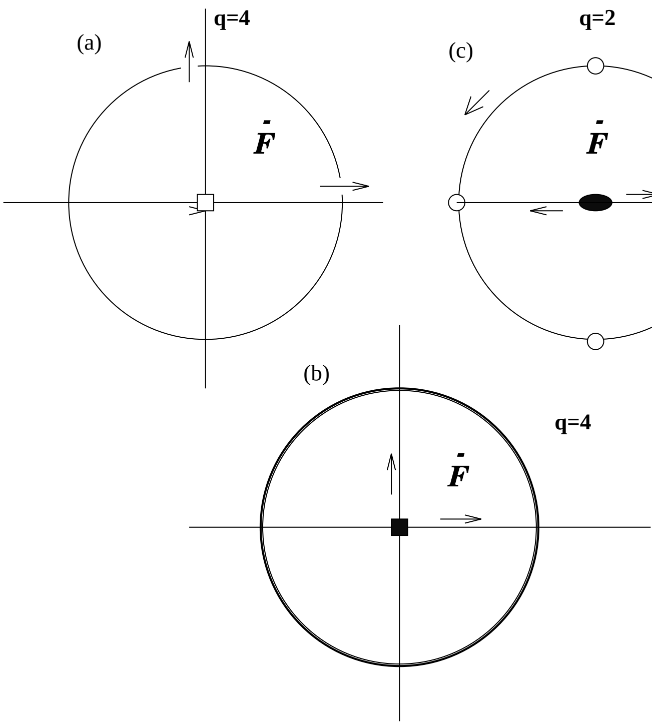

FIGURE 3. Spherical groups for $\mathbf{S^2} \times \mathbf{R}$ space forms
(a) $\mathbf{1q} - \mathbf{q}, \mathbf{q}$ for **Or1** (b) $\mathbf{5q} - \mathbf{q}*$ for **No1** and **No2**
(c) $\mathbf{7q} - \mathbf{q}\otimes$ for **No1**, **No2**, and **Or2**, respectively.

group G by (2.2) shall be fixed point free.
We recall from [1] the three types of $\mathbf{S^2}\times\mathbf{R}$ point groups derived from any isometry group A of $\mathbf{S^2}$:

$$\text{(3.2)} \qquad \textit{Type I}: G_0 = A \times 1_{\mathbf{R}}, \ \textit{Type II}: G_0 = A \times \overline{1}_{\mathbf{R}}$$

$$\textit{Type III}: G_0 = A'B := \{B \times 1_{\mathbf{R}}\} \cup \{(A \setminus B) \times \overline{1}_{\mathbf{R}}\}$$

where B is a subgroup in A of index two.

Type I: $(\mathbf{q},\mathbf{q}) \times 1_{\mathbf{R}}$ from (3.1) has the presentation

$$\text{(3.3)} \qquad (g_1 - g_1^q) \ g_1 \in \mathbf{S^2_2}$$

with one generator $g_1 := \mathbf{r} \times 1_{\mathbf{R}}$ and relation $g_1^q = 1$. The possible translational part k_1 in (g_1, k_1) satisfies, by (2.4), the so called Frobenius congruence

$$\text{(3.4)} \qquad k_1 q \equiv 0 \pmod 1$$

implying $k_1 \equiv 0$ or $k_1 \equiv \frac{k}{q}$, $k = 1, \dots, q-1$.
The first solution leads to fixed point free group iff $q = 1$, the second ones make this if $(k, q) = 1$, just as we have described in Sect.2 (in Rem. 4, formula (2.11), Prop.2.3).

Type II: $(\mathbf{q},\mathbf{q}) \times \overline{1}_{\mathbf{R}}$ from (3.1) has the presentation

$$\text{(3.5)} \qquad (g_1, g_2 - g_1^q, g_2^2, g_1^{-1} g_2 g_1 g_2), \ g_1 \in \mathbf{S^2_2}, \ g_2 \in \mathbf{R_1}$$

for $g_1 := \mathbf{r} \times 1_{\mathbf{R}}$ and $g_2 : (\varphi, \vartheta, y) \mapsto (\varphi, \vartheta, -y)$. The translational parts k_1 and k_2 in (g_1, k_1) and (g_1, k_2) satisfy the Frobenius congruences

$$\text{(3.6)} \qquad k_1 q \equiv 0, \ k_2 2 \equiv 0, \ k_1 2 \equiv 0 \pmod 1.$$

Now we have only to emphasize that for any $k_2 \in \mathbf{R}$

$$\text{(3.7)} \qquad (g_2, k_2) : (\varphi, \vartheta, y) \mapsto (\varphi, \vartheta, -y + k_2) \in \mathbf{R_1}$$

is a reflection in the $\mathbf{S^2} \times \{\frac{1}{2}k_2\}$ level with fixed points.
Thus we do not obtain any space form group in the *Type II*.

Type III: $(\mathbf{q},\mathbf{q})'(\frac{\mathbf{q}}{\mathbf{2}}, \frac{\mathbf{q}}{\mathbf{2}})$ with $2 \le q$ even yields a presentation

$$\text{(3.8)} \qquad (g_1 - g_1^q), \ g_1 : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{2\pi}{q}, \vartheta, -y) \in \mathbf{S^2_2 R_1}$$

Any transform (g_1, k_1), $k_1 \in \mathbf{R}$ for any even $q \ge 2$, has fixed points: $(., \frac{\pi}{2}, \frac{k_1}{2})$, $(., -\frac{\pi}{2}, \frac{k_1}{2})$ over the poles of $\mathbf{S^2}$, yielding no space form in this type.■

The next important isometry group series of $\mathbf{S^2}$ (Table 1) is

$$\text{(3.9)} \qquad \mathbf{7q} - (-, 1; [q]; \{\}), \ q \ge 1 \ - \ \mathbf{q}\otimes.$$

Here every 2-orbifold $(\mathbf{S^2}/A)$ is nonorientable $(-)$ surface with genus 1 (i.e. a projective plane, or i.e., the sphere with one cross cap $\otimes$) with a rotation centre of order q (cone point with angular neighborhood $\frac{2\pi}{q}$), in Fig.3 we have pictured its symbolic fundamental domain $\overline{\mathcal{F}}_{\mathbf{q}\otimes}$ with its side pairing. This provides the generator

$$\mathbf{z} : (\varphi, \vartheta) \mapsto (\varphi + \frac{\pi}{q}, -\vartheta) \in \mathbf{S^2_3} \tag{3.10}$$

a rotatory reflection of $\mathbf{S^2}$.
The possible point groups as follow:

Type I: $\mathbf{q}\otimes \times 1_{\mathbf{R}}$ has the presentation

$$(g_1 - g_1^{2q}),\ g_1 \in \mathbf{S^2_3}\ :\ g_1 : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{\pi}{q}, -\vartheta, y), \tag{3.11}$$

k_1 in (g_1, k_1) satisfies the Frobenius congruence

$$k_1 2q \equiv 0 \pmod 1\ :\ k_1 \equiv 0;\ k_1 \equiv \frac{1}{2};\ k_1 \equiv \frac{k}{2q},\ k = 1, 2, \ldots, q-1. \tag{3.12}$$

The diffeomorphism class **No1** of nonorientable $\mathbf{S^2} \times \mathbf{R}$ space forms will be represented by $\mathbf{7}, \mathbf{1.I.1}(0)$ from [1], i.e. in case $q = 1$, $k_1 = 0$, $\mathbf{z} := g_1$ (Fig.4). The fundamental domain $\overline{\mathcal{F}}_G$ of this G is a "half shell" with unusual face pairing which provides the presentaion(by unusual "edges")

$$G = (\mathbf{z}, \tau - \mathbf{z}^2, \mathbf{z}\tau\mathbf{z}\tau^{-1})\ \sim \mathbf{Z_2} \times \mathbf{Z}. \tag{3.13}$$

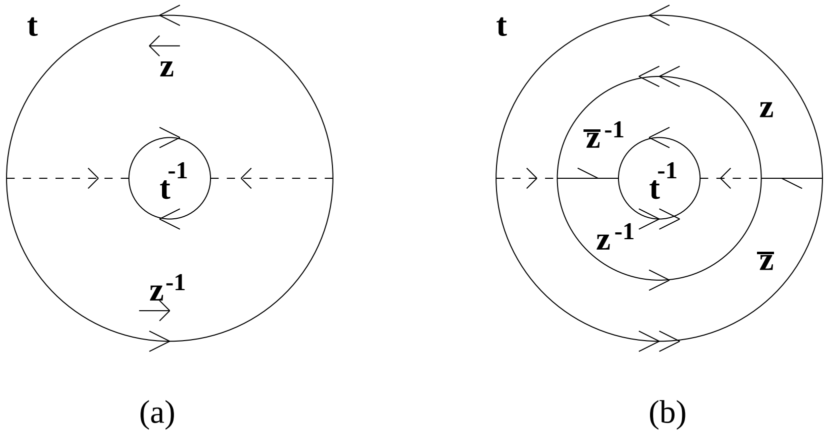

FIGURE 4. Non-orientable $\mathbf{S^2} \times \mathbf{R}$ space forms by Schlegel diagrams of half shells $\bar{\mathcal{F}}$ with side pairings
(a) $\mathbf{No1}(\mathbf{Z_2} \times \mathbf{Z})$: $\langle \mathbf{z} \rangle \times \langle \tau \rangle$ is generated by the antipodal map $\mathbf{z} \in \mathbf{S^2_3}$ and by a translation $\tau \in \mathbf{R_2}$ with relations to the "edges" $\rightarrow$: $\mathbf{z}\tau\mathbf{z}^{-1}\tau^{-1} = 1$; $\dashrightarrow$: $\mathbf{z}\mathbf{z} = 1$
(b) $\mathbf{No2}(\mathbf{Z})$: $\langle \mathbf{z} \rangle$, $\mathbf{z} \in \mathbf{S^2_3 R_2}, \tau \in \mathbf{R_2}$, $\bar{\mathbf{z}} \in \mathbf{S^2_3 R_2}$ $\dashrightarrow$: $z\bar{z}^{-1} = 1$, $\rightharpoonup$: $\bar{\mathbf{z}}^{-1}\mathbf{z} = 1$, $\twoheadrightarrow$: $\mathbf{z}\bar{\mathbf{z}}\tau^{-1} = 1$, $\rightarrow$: $\bar{z}z\tau^{-1} = 1$

The second diffeomorphism class **No2** of nonorientable $\mathbf{S^2} \times \mathbf{R}$ space forms will be represented by $\mathbf{7}, \mathbf{1.I.2}(\frac{1}{2})$ from [1], i.e. in case $q = 1$, $k_1 = \frac{1}{2}$ (Fig.4). The fundamental domain $\bar{\mathcal{F}}_G$ is again a "half shell" with another face pairing with presentation

$$G = (\mathbf{z}, \overline{\mathbf{z}}, \tau - \mathbf{z}\overline{\mathbf{z}}\tau^{-1}, \overline{\mathbf{z}}\mathbf{z}\tau^{-1}, \mathbf{z}\overline{\mathbf{z}}^{-1}, \overline{\mathbf{z}}^{-1}\mathbf{z}) \sim \mathbf{Z}, \tag{3.14}$$

since $\overline{\mathbf{z}} = \mathbf{z}$, $\tau = \mathbf{zz}$ are consequences. Other $q > 1$ leads to fixed points over the poles of $\mathbf{S^2}$ in both above cases $k_1 = 0$ or $k_1 = \frac{1}{2}$.

The *third case* in (3.12) yields fixed point free group iff the g.c.d $(k, q) = 1$. Then the generator of the group G

$$\mathbf{z} := (g_1, k_1) : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{\pi}{q}, -\vartheta, y + \frac{k}{2q}) \tag{3.15}$$

leads to cases: i, q odd, k even ii, q odd, k odd iii, q even.

i, $1 < q$ odd, $k = 2u$, $1 \leq u \in \mathbf{N}$. Consider the element

$$\mathbf{w} := \mathbf{z}^q \tau^{-u} : (\varphi, \vartheta, y) \mapsto (\varphi + \pi, -\vartheta, y) \in \mathbf{S_3^2} \tag{3.16}$$

which is just the antipodal map of $\mathbf{S^2}$, an orientation reversing involution, i.e. $\mathbf{ww} = 1$. The following element $\overline{\mathbf{z}}$ - with t, v odd - will be

$$\overline{\mathbf{z}} := \mathbf{z}^v \tau^{-t} : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{v\pi}{q}, -\vartheta, y + \frac{1}{q}) \in \mathbf{S_3^2 R_2}, \tag{3.17}$$

here $2uv - 2tq = 2$, i.e. $uv - tq = 1$, because of g.c.d. $(u, q) = 1$ can be chosen. This provides a minimal (non zero) translational part, uniquely, since different v_1, v_2 (mod q) could not serve this translational part. Then

$$G = \langle \mathbf{w} \rangle \times \langle \mathbf{w}\overline{\mathbf{z}} \rangle \sim \mathbf{Z_2} \times \mathbf{Z}, \tag{3.18}$$

and the skew transform S by (2.8) with $\alpha = \frac{(v+q)\pi}{q}$, $a = \frac{1}{q}$ shows that $\mathbf{S^2} \times \mathbf{R}/G$ belongs to the diffeomorphism class **No1** by (3.13). To this, following Prop.2.2, we can check with $\mathbf{z}$ in (3.13) that

$$\mathbf{w} = S^{-1}\mathbf{z}S, \ \mathbf{w}\overline{\mathbf{z}} = S^{-1}\tau S \tag{3.19}$$

hold, indeed.

In cases ii, and iii, $\mathbf{z}$ in (3.15) does not produce an involutive element of G. With appropriate integers t, v odd we take

$$\overline{\mathbf{z}} := \mathbf{z}^v \tau^{-t} : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{v\pi}{q}, -\vartheta, y + \frac{1}{2q}) \in \mathbf{S_3^2 R_2} \tag{3.20}$$

where $kv - 2qt = 1$ since g.c.d $(k, 2q) = 1$.
This $\overline{\mathbf{z}}$ provides a minimal (non zero) translational part, uniquely, since different v_1, v_2 (mod q) could not serve this translational part (we may apply also the similarity (2.10)). Then

$$G = \langle \overline{\mathbf{z}} \rangle \sim \mathbf{Z} \tag{3.21}$$

leads to the diffeomorphism class **No2** by (3.14), again by the skew transform S in (2.8).

Type II: $\mathbf{q}\otimes \times \bar{1}_\mathbf{R}$ leads to fixed points analogously as before. We do not obtain any $\mathbf{S^2} \times \mathbf{R}$ space form.

Type III: $(\mathbf{q}\otimes)'(\mathbf{q},\mathbf{q})$ has the presentation

$$(3.22) \qquad (g_1 - g_1^{2q}),\ g_1 : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{\pi}{q}, -\vartheta, -y) \in \mathbf{S_3^2 R_1}.$$

The translational part k_1 in (g_1, k_1) satisfies by (2.4)

$$(3.23) \qquad k_1 \mapsto (-k_1) + k_1 = 0 \mapsto k_1 \ldots \mapsto 0.$$

Thus, by choosing the similarity $\varrho : (\varphi, \vartheta, y) \to (\varphi, \vartheta, y + \frac{1}{2}k_1)$ (as translation), we get equivariance to case $k_1 = 0$. We use the notation $\mathbf{f} := g_1$ for this involutive transform in case $q = 1$, which is the product of the antipodal map of $\mathbf{S^2}$ and a reflection in $\mathbf{S^2} \times \{0\}$. Else $(q > 1)$ we obtain fixed points over the poles of $\mathbf{S^2}$. Thus we get the promised representative $\mathbf{S^2} \times \mathbf{R}/G$ for the second diffeomorphism class **Or2** of orientable $\mathbf{S^2} \times \mathbf{R}$ space forms in Fig.1 with

$$(3.24) \qquad G := (\mathbf{f}, \tau - \mathbf{f}^2, \mathbf{f}\tau\mathbf{f}\tau)$$

by half shell $\bar{\mathcal{F}}_2$. Or equivalently $G := (\mathbf{f}_1, \mathbf{f}_2 - \mathbf{f}_1^2, \mathbf{f}_2^2)$ holds by a shell $\bar{\mathcal{F}}_1$, if $\mathbf{f}_1 := \mathbf{f}$, and $\mathbf{f}_2 := \mathbf{f}\tau : (\varphi, \vartheta, y) \mapsto (\varphi + \pi, -\vartheta, -y + 1)$ for the free Coxeter group $\langle \mathbf{f}_1 \rangle \times \langle \mathbf{f}_2 \rangle$. This was summarized at the end of Sect.2.

4. The other similarity classes of $\mathbf{S^2} \times \mathbf{R}$ space forms

After the discussions detailed before, we treat the remaining cases more sketchily. From the finite isometry groups of $\mathbf{S^2}$ only (Fig.3)

$$(4.1) \qquad \mathbf{5q} - (+, 0; [q]; \{(1)\}),\ 1 \le q \in \mathbf{N}\ -\ \mathbf{q}*$$

provide $\mathbf{S^2} \times \mathbf{R}$ space forms. From the other spherical groups in Table 1 each leads to fixed points as in [1]. So the classification of orientable space forms has already been complete.

Any group A in (4.1) are generated by a reflection g_1 in an equatorial circle of $\mathbf{S^2}$ and by a q-fold rotation g_2 about its poles. The fundamental domain $\bar{\mathcal{F}}_A$ in Fig.3 shows also the $\mathbf{S^2}$-orbifold with one boundary component in $\{(1)\}$ or the empty sign after $*$, i.e. without non trivial dihedral corner; moreover, if $q > 1$, one q-fold rotation centre (cone point of angular neighborhoud $\frac{2\pi}{q}$) in $[q]$ or q before $*$, respectively.

Again, we consider the possible 3 types of point groups G_0 and the corresponding $\mathbf{S^2} \times \mathbf{R}$ space groups G without any fixed point.

Type I: $\mathbf{q}* \times 1_\mathbf{R}$ has the presentation

$$(4.2) \qquad (g_1, g_2 - g_1^2, g_2^q, g_2^{-1} g_1 g_2 g_1),$$

$$g_1 : (\varphi, \vartheta, y) \mapsto (\varphi, -\vartheta, y);\ g_2 : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{2\pi}{q}, \vartheta, y).$$

The translational parts k_1 in (g_1, k_1) and k_2 in (g_2, k_2) satisfy the Frobenius congruences

$$k_1 2 \equiv 0,\ k_2 q \equiv 0,\ -k_2 + k_1 + k_2 + k_1 = 2k_1 \equiv 0 \pmod 1. \tag{4.3}$$

Only $(k_1, k_2) = (\frac{1}{2}, \frac{k}{q})$ provides nonorientable $\mathbf{S^2} \times \mathbf{R}$ space forms. Then

$$\mathbf{a} := (g_1, \frac{1}{2}) : (\varphi, \vartheta, y) \mapsto (\varphi, -\vartheta, y + \frac{1}{2}) \tag{4.4}$$

$$\mathbf{s} := (g_2, \frac{k}{q}) : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{2\pi}{q}, \vartheta, y + \frac{k}{q})$$

with g.c.d $(k, q) = 1$ and $k = 1, \dots, \lfloor \frac{q}{2} \rfloor$ for l.i.p of $\frac{q}{2}$, generate our group G. We discuss the two cases: i, $q = 2u$ even and ii, q odd.

i, If $q = 2u$ is even, we take

$$\mathbf{w} := \mathbf{a}\mathbf{s}^v \tau^{-t} : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{2\pi v}{2u}, -\vartheta, y) \in \mathbf{S_3^2} \tag{4.5}$$

by $\frac{1}{2} + \frac{kv}{2u} - t = 0$, i.e. $2u(1 - 2t) + 2kv = 0$ holds with $v = u = \frac{q}{2}$, $k = 2t - 1$, $\mathbf{ww} = 1$, thus $\mathbf{w}$ is the involutive antipodal map. The transform

$$\overline{\mathbf{s}}\ :\ \mathbf{s}^s \tau^{-r}\ :\ (\varphi, \vartheta, y) \mapsto (\varphi + \frac{\pi s}{u}, \vartheta, y + \frac{1}{q}) \in \mathbf{S_2^2 R_2}, \tag{4.6}$$

where

$$\frac{ks}{2u} - r = \frac{1}{2u},\ \textit{i.e. } ks - 2ur = 1 \textit{ since g.c.d } (k, 2u) = 1,$$

is just the unique screw motion in G with minimal non-zero translational part. We see, that

$$G = \langle \mathbf{w} \rangle \times \langle \overline{\mathbf{s}} \rangle \sim \mathbf{Z_2} \times \mathbf{Z} \tag{4.7}$$

serves an $\mathbf{S^2} \times \mathbf{R}$ space form diffeomorphic to **No1** by a skew transform S by (2.8) as earlier. To this

$$\mathbf{a}\ := \mathbf{w}\tau^t \mathbf{s}^{-v} = \mathbf{w}\overline{\mathbf{s}}^{(qt - kv)} = \mathbf{w}\overline{\mathbf{s}}^u, \tag{4.8}$$

$$\mathbf{s} = \overline{\mathbf{s}}^k$$

by (4.5) and (4.6) are satisfactory equations, according to (4.4).

ii, If q is odd, then we can take with integer s and t

$$\overline{\mathbf{z}} :\ \mathbf{a}\mathbf{s}^s \tau^{-t} : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{2\pi s}{q}, -\vartheta, y + \frac{1}{2q}) \in \mathbf{S_3^2 R_2} \tag{4.9}$$

by

$$\frac{1}{2} + \frac{sk}{q} - t = \frac{1}{2q}\ ,\ \textit{i.e. } 2sk + (1 - 2t)q = 1 \textit{ by g.c.d } (2k, q) = 1,$$

as a unique generator. Moreover, G does not have any involutive element now. Namely, we can express the generators in (4.4) by $\overline{\mathbf{z}}$:

$$\mathbf{a} = \overline{\mathbf{z}}^q,\ \mathbf{s} = \overline{z}^{2k} \tag{4.10}$$

by (4.9). This proves $G = \langle \overline{\mathbf{z}} \rangle \sim \mathbf{Z}$, and a skew transform S by (2.8) shows the diffeomorphic equivariance to G in (3.20-21) in the class **No2**.

Type II: $\mathbf{q}* \times \overline{1}_{\mathbf{R}}$ leads to fixed points as before.

Type III: We have three possibilities as in [1]:

$$\text{(4.11)} \qquad \mathbf{III.a}\ (\mathbf{q}*)'(\mathbf{q},\mathbf{q}),\ \mathbf{III.b}\ (\mathbf{q}*)'(\frac{\mathbf{q}}{\mathbf{2}}*),\ \mathbf{III.c}(\mathbf{q}*)'(\frac{\mathbf{q}}{\mathbf{2}}\otimes),$$

where $\mathbf{q}$ is even in the last two cases. Again, we have fixed points at each space group G from the above three point groups.

Remarks 1, If we do not require an invariant lattice $\langle \tau \rangle$ by (2.1), then we have uncountably many similarity classes in **No2** as well by a generator

$$\text{(4.12)} \qquad \overline{\mathbf{z}} : (\varphi, \vartheta, y) \mapsto (\varphi + \alpha, -\vartheta, y + a) \in \mathbf{S^2_3 R_2}$$

for G with irrational $\frac{\alpha}{2\pi} \in (0, \frac{1}{2});\ 0 < a \in \mathbf{R}$.
This is as to in the orientable class **Or1** with the screw motion $\mathbf{s}$ in (2.6).
2, In the orientable case $\mathbf{f} \in \mathbf{S^2_3 R_1}$ by (2.13) is the only (even similarity) type of involutive transforms without fixed points. Combining this $\mathbf{f}$ with a screw motion $\mathbf{s}$ in (2.6) with irrational $\frac{\alpha}{2\pi}$

$$\text{(4.13)} \qquad \mathbf{fsfs} : (\varphi, \vartheta, y) \mapsto (\varphi + 2\pi + 2\alpha, \vartheta, y)$$

has fixed points over the poles. Thus we see that the diffeomorphism class **Or2** has exactly one similarity class of $\mathbf{S^2 \times R}$ space forms, and we do not have any more.
3, In the nonorientable case the antipodal map $\mathbf{z} \in \mathbf{S^2_3}$ in (3.13) is the only type of involutive transforms without fixed points. Combining this $\mathbf{z}$ with any screw motion $\mathbf{s}$ in (2.6)

$$\text{(4.14)} \qquad \mathbf{zsz} : (\varphi, \vartheta, y) \mapsto (\varphi + 2\pi + \alpha, \vartheta, y + a), \mathbf{zsz} = \mathbf{s}$$

show that our diffeomorphism class **No1** contains also uncountably many similarity classes, and we do not have any more.

At the end we summarize our results in

Theorem 4.1 *There are exactly 2 orientable:* $\mathbf{Or1(Z)}$ *and* $\mathbf{Or2(Z \otimes Z)}$, *resp. 2 nonorientable diffeomorphism classes:* $\mathbf{No1(Z_2 \times Z)}$ *and* $\mathbf{No2(Z)}$ *of* $\mathbf{S^2 \times R}$ *space forms, containing the similarity equivariance classes as follows in Table 2. In the diffeomorphism class* **Or2** *there are exactly one similarity type, also in the general sense if we allow infinite point groups for the fundamental groups. The other 3 diffeomorphism classes contain similarity classes in infinite series for finite point groups, or uncountably many similarity classes for infinite point groups.*□

Symbol	Conditions	Diffeomorphism class
1,1.I.1(0)	representative (Fig.1)	$\mathbf{Or1}(\mathbf{Z})$
1q.I.2($\frac{k}{q}$)	$(k,q)=1,\ 1\le k\le\lfloor\frac{q}{2}\rfloor$	**Or1**
7,1.I.1(0)	representative (Fig.4)	$\mathbf{No1}(\mathbf{Z_2}\times\mathbf{Z})$
7,1.I.2($\frac{1}{2}$)	repr. (Fig.4)	$\mathbf{No2}(\mathbf{Z})$
7qo.I.3($\frac{k}{2q}$)	$2\le q$ odd, $(k,q)=1$, even $k<q$	**No1**
7qo.I.3($\frac{k}{2q}$)	$2\le q$ odd, $(k,q)=1$, odd $k<q$	**No2**
7qe.I.3($\frac{k}{2q}$)	$2\le q$ even, $(k,q)=1$, $k<q$	**No2**
7,1.III.1(0)	repr. (Fig.1)	$\mathbf{Or2}(\mathbf{Z_2}\otimes\mathbf{Z_2})$
5qe.I.4($\frac{1}{2},\frac{k}{q}$)	$2\le q$ even, $(k,q)=1$, $1\le k\le\lfloor\frac{q}{2}\rfloor$	**No1**
5qo.I.4($\frac{1}{2},\frac{k}{q}$)	$1\le q$ odd, $(k,q)=1$, $1\le k\le\lfloor\frac{q}{2}\rfloor$	**No2**

Table 2 Classes of $\mathbf{S^2}\times\mathbf{R}$ space forms

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