SIMILARITY AND DIFFEOMORPHISM CLASSIFICATION OF $S^2 \times \mathbb{R}$ MANIFOLDS

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1. INTRODUCTION

The 3-space $S^2 \times \mathbb{R}$ is the direct product of the 2-sphere and the real line. The similarity group

$$\text{Sim}(S^2 \times \mathbb{R}) := \text{Isom}(S^2) \times \text{Sim}(\mathbb{R}) := \{A \times \{(a,b)\}$$

where $A \in O^3$ the 3-dimensional orthogonal group acting on $S^2$; $a \in \mathbb{R}\setminus\{0\}$, $b \in \mathbb{R}$ and $x \mapsto xa + b$ define a similarity of $\mathbb{R}$.

The isometry group

$$\text{Isom}(S^2 \times \mathbb{R}) := \text{Isom}(S^2) \times \text{Isom}(\mathbb{R})$$

is specified by $a := \pm 1$.

At the similarity classification of $S^2 \times \mathbb{R}$ space groups in [1], the fixed point free isometry groups $G$, leaving invariant a translation lattice of $\mathbb{R}$, have also been found and listed in infinite series which lead to space forms $S^2 \times \mathbb{R}/G$, i.e. compact manifolds with local $S^2 \times \mathbb{R}$ metric [2],[3],[4] (see our Table 2).

It turns out that - instead of similarity equivariance - the diffeomorphism one

$$G \sim G = S^{-1}GS$$

with a very simple “skew” diffeomorphism $S$ leads to 4 diffeomorphism classes of $S^2 \times \mathbb{R}$ space forms derived first very sketchily in [5]:

2 orientable ones (with fundamental group $\mathbb{Z}$ and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$, respectively; here $\otimes$ stands for free product of Coxeter groups)

and 2 nonorientable ones (with $\mathbb{Z}_2 \times \mathbb{Z}$ and $\mathbb{Z}$, respectively).

Surprisingly, we find in the book [4] - without any proof - the statement on the existence of one nonorientable manifold, up to diffeomorphism, that admits $S^2 \times \mathbb{R}$ structures. This statement is false then obviously, in the earlier survey [3] we can read the correct numbers.

We are working - in this comparison - on the classification of space forms in the other fibre geometries $H^2 \times \mathbb{R}$, $SL_2 \mathbb{R}$ and Nil as well.

Although P. Scott [3] has presented a strategy for describing all the Seifert bundles for the four compact $S^2 \times \mathbb{R}$ manifolds, we find it actual to give another

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more complete interpretation which seems to be advantageous for other reason (see also [1] and [2]).

2. $S^2 \times \mathbb{R}$ ISOMETRIES AND SPACE FORMS, BASIC OBSERVATIONS

As we mentioned in the Introduction, an $S^2 \times \mathbb{R}$ space form can naturally be defined as a factor space $(S^2 \times \mathbb{R})/G$, where $G$ is an isometry group of $S^2 \times \mathbb{R}$, containing an invariant lattice in $\mathbb{R}$, denoted by $L_G$, as follows

$$G \triangleright L_G = \langle \tau \rangle, \tau : S^2 \times \mathbb{R} \rightarrow S^2 \times \mathbb{R}, (X,x) \mapsto (X,x+t)$$

with a minimal $0 < t \in \mathbb{R}$; moreover, $G$ acts freely on $S^2 \times \mathbb{R}$ (i.e. without any fixed point) with compact fundamental domain (of non-empty interior).

By a similarity of $S^2 \times \mathbb{R}$ we may assume that $t = 1$. $G$ is called space form group or fundamental group as well.

$$G := \{ A_i \times k_i \} := \{ A_i \times (K_i,k_i) \} := \{ A_i \times K_i, k_i \}$$

where $A_i \in O^3$ acts on $S^2$, $k = (K_i,k_i)$ acts on $\mathbb{R}$. Here $K_i$ is either the identity $1_\mathbb{R}$ of $\mathbb{R}$ or the reflection in zero $1_{-\mathbb{R}} : x \mapsto -x$. The "linear parts" of $G$ in (2.2) form the point group

$$G_0 = \{ (A_i \times K_i) \}$$

of $G$. The translational parts $k_i$ to $(A_i \times K_i)$ have to satisfy the multiplication formula

$$(A_1 \times K_1, k_1) \circ (A_2 \times K_2, k_2) = (A_1 A_2 \times K_1 K_2, k_1 K_2 + k_2)$$

where we have indicated that our transforms act from the right throughout this paper. Formula (2.4) can be derived from the assumed right action, in general:

$$(X,x)(A_i \times k_i) = (XA_i,x K_i + k_i).$$

Any isometry of $S^2 \times \mathbb{R}$ is a product of at most 5 reflections. At most 3 reflections (in equator circles of $S^2$) produce any element of Isom$S^2$—Isom$S^2 \times \text{Id} \mathbb{R}$, at most 2 reflections (in points of $\mathbb{R}$) are for Isom$\mathbb{R}$—Id$S^2 \times \text{Isom} \mathbb{R}$.

$S^2_{R_i}$ denotes the set of reflections above, where $i = 0 \ldots 3$, $j = 0 \ldots 2$ (respectively, $i = 0$ and $j = 0$ for Id($S^2 \times \mathbb{R}$)).

**Proposition 2.1** Any space form group $G$ has a finite point group $G_0$.

The proof is indirect. Since the linear parts of Isom$\mathbb{R}$ contain 2 elements, then $\{ A_i \}$ in (2.3) would have infinitely many ones from Isom$S^2$. But $S^2$ is compact, and we assumed a lattice $L_G = \langle \tau \rangle \triangleleft G$. Thus, there does not exist any open set in the compact "shell" $S^2 \times [0,1]$ (Fig.1) which contains only points not equivalent under the infinitely many transforms $\{ (A_i \times 1_\mathbb{R}, k_i), 0 \leq k_i < 1 \} =: G_0 \subset G$.

Then $G$ cannot have any fundamental domain with non-empty interior $F_G$, since the infinite disjoint union of $G_0$-images of this $F_G$ would lie in the compact shell $S^2 \times [0,2]$, a contradiction. $\blacksquare$

**Remarks 1.** In the proof we did not utilize, that $G$ was fixed point free.

2. If $G$ is not assumed to have a lattice, then it may have infinite point group $G_0$. 
SIMILARITY AND DIFFEOMORPHISM CLASSIFICATION OF $S^2 \times \mathbb{R}$ MANIFOLDS

Figure 1. $S^2 \times \mathbb{R}$ is modelled in $E^3_\infty := E^3 \cup \{\infty\}$ where the origin 0 and the infinity $\infty$ are distinguished. The 0-concentric sphere of Euclidean radius $r$ models the level $S^2 \times \{r\}$ by $r = \ln x$. Thus 0 is a joint point $-\infty$ of the $\mathbb{R}$-fibres $\{s\} \times \mathbb{R}$ ($s \in S^2$) as 0-rays, $\infty$ is a common point $+\infty \in \{s\} \times \mathbb{R}$. The spherical transforms are usual. The transforms of $\mathbb{R}$ appear as the following “dictionary” translates:

- reflection ($\in \mathbb{R}_1$) of $S^2 \times \mathbb{R} \iff$ sphere inversion of $E^3_\infty$ in a sphere $S^2 \times \{k\}$ in an 0-centered sphere of radius $\varphi$ where $k = \ln \varphi$
- translation ($\in \mathbb{R}_2$) of $S^2 \times \mathbb{R} \iff$ 0-central similarity of $E^3_\infty$ with $d \in \mathbb{R}$ from 0 with factor $\lambda$ where $d = \ln \lambda$

(a) $\mathcal{F}$ is a shell describing $\text{Or}1(Z)$, generated by a translation $\tau$ pairing the spheres $S_{i-1}$ and $S_i$ of $\mathcal{F}$ (the letter $S$ is left in the figure).

(b) $\text{Or}2(Z_2 \otimes Z_2)$ is represented by the shell $\mathcal{F}_1$, each of its boundary spheres is paired with itself by an involutive map $f_i \in S^3_2 \mathbb{R}_i$ ($i = 1, 2$).

(c) Or equivalently, a half shell $\tilde{\mathcal{F}}_2$ and its Schlegel diagram in picture (d) describes $\text{Or}2$ by $(f, \tau = ff, f\tau f^{-1})$.
With \( y \in \mathbb{R} \) and with the usual (geographic) sphere coordinates \( \varphi \) (mod \( 2\pi \)) and 
\[-\frac{\pi}{2} \leq \vartheta \leq \frac{\pi}{2}, \]
any “screw motion” of \( S^2 \times \mathbb{R} \)

\[
(2.6) \quad s : (\varphi, \vartheta, y) \mapsto (\varphi + \alpha, \vartheta, y + a); \quad \frac{\alpha}{2\pi} \in \mathbb{Q}^*; \quad 0 < a \in \mathbb{R}
\]
generates a cyclic group \( G := \langle s \rangle \) with infinite point group \( G_0 \) (\( \mathbb{Q}^* \) denotes the set of irrational numbers). The orbit space \( S^2 \times \mathbb{R}/\langle s \rangle \) can be represented by the “shell-like” compact fundamental domain \( \mathcal{F} = S^2 \times [0, a] \) with a pairing (the bar refers to this) of its \( 0 \)- and \( a \)-level by (2.6). See Fig.1 for an analogous picture.

\( G \) is fixed point free, i.e. we get a compact manifold with local \( S^2 \times \mathbb{R} \)-metric. Then

\[
(2.7) \quad S^2 \times \mathbb{R}/\langle s \rangle \sim \mathcal{F}
\]

may be called an \( S^2 \times \mathbb{R} \) space form in general sense. Then we promptly have uncountable many similarity classes of \( S^2 \times \mathbb{R} \) space forms, parametrized just by the irrational number \( \alpha/2\pi \in (0, 1/2) \). The similarity parameter \( a \) in (2.6) is not essential.

As we have promised in the introduction, we can formulate the illustrative

**Proposition 2.2** Any \( S^2 \times \mathbb{R}/\langle s \rangle \) above is diffeomorphic to \( S^2 \times \mathbb{R}/\langle \tau \rangle \), in (2.1) with \( t = 1 \) by the “skew” transform

\[
(2.8) \quad S : S^2 \times \mathbb{R} \to S^2 \times \mathbb{R} : (\varphi, \vartheta, y) \mapsto (\varphi + \alpha, \vartheta, y + a)
\]

so that \( s = S^{-1} \tau S \).

*Proof* (see the symbolic Fig.2). By our conventions for the coordinates of \( S^2 \times \mathbb{R} \)
and for the parameters of \( s \) in (2.6), the skew transform \( S \) is a bijection, indeed. For this \( \varphi \leftrightarrow y \), \( \vartheta \leftrightarrow \varrho \) are obvious. If \( \varphi \) runs over an interval of length \( 2\pi \), then so does \( \varphi = \varphi + \varrho \) for any fixed \( \varrho \). Moreover, the Jacobian

\[
(2.8') \quad \frac{\partial(\varphi, \vartheta, y)}{\partial(\varphi, \varrho, \varrho)} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}
\]

is constant.

Since \( \tau : (\varphi, \vartheta, \varrho) \mapsto (\varphi, \vartheta, \varrho + 1) \) is a unit translation, thus

\[
(2.9) \quad (\varphi, \varrho, y) S^{-1}_0 (\varphi, \vartheta, \varrho) = (\varphi, \varrho, y + 1) S_0 (\varphi + (\varrho + 1) \alpha, \vartheta, (\varrho + 1) \alpha) = (\varphi + \alpha, \vartheta, y + a) \quad \text{as at } s.
\]

**Remarks 3.** As before we can see that \( s \) in (2.6) is similarity equivariant to

\[
(2.10) \quad \sigma : (\varphi, \vartheta, \varrho) \mapsto (\varphi, \vartheta, y) := (-\varphi, \vartheta, y + a);
\]

holds indeed.

Thus we have proven all statements in Rem. 2.
4. The screw motion, with \(2 \leq q \in \mathbb{N}\) (for natural numbers)

\[
(2.11) \quad s : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{2\pi}{q}, \vartheta, y + \frac{k}{q}) \in S^2 \times \mathbb{R}
\]

with greatest common divisor \((g, d) (k, q) = 1\), and \(1 \leq k \leq \lfloor \frac{q}{2} \rfloor\) (the lower integer part of \(\frac{q}{2}\)) and the lattice \((\tau)\) in (2.1) with \(t = 1\), determine an orientable space form \(S^2 \times \mathbb{R}/G\) in our original (restricted) definition. These lie in different similarity classes for different pairs \(q, k\) above. However, they are all diffeomorphic to \(S^2 \times \mathbb{R}/(\tau)\) by Prop. 2.2, so with the cyclic fundamental group \(G \sim \mathbb{Z}\). To this we consider the transform

\[
(2.12) \quad s^t : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{2\pi u}{q}, \vartheta, y + \frac{k_u}{q} - v)
\]

from \(G\), where \(k_u - qv = 1\) can be achieved, since \((k, q) = 1\), by appropriate integers \(u, v\) with \(0 < u < q\) and \(0 \leq v < k\). Different \(k_1\) and \(k_2\) cannot yield the same \(u\) in (2.12), else \(q\) would divide \(u\), a contradiction. However, \(k\) and \(q - k\) lead to equivariant groups by similarity of type (2.10).■

The diffeomorphism class, represented by \(S^2 \times \mathbb{R}/(\tau)\) by Prop. 2.2 will be denoted by \(\text{Or}_1(\mathbb{Z})\). We summarize the previous results in

**Proposition 2.3** The diffeomorphism class \(\text{Or}_1(\mathbb{Z})\) of \(S^2 \times \mathbb{R}\) space forms contains the infinite series of similarity classes described exactly in Rem.4., formula (2.11).

The proof is completed by observing the angular invariant \(\alpha = \frac{2\pi u}{q} = -\frac{2\pi (q - u)}{q}\) (mod \(2\pi\)) belonging to the shortest translation part of length \(\frac{1}{q}\) in (2.12).

Moreover, we shall find \(\text{Or}_2(\mathbb{Z}_2 \oplus \mathbb{Z}_2)\) as a diffeomorphism class, containing exactly one similarity class of the remaining orientation preserving fixed point free isometry groups of \(S^2 \times \mathbb{R}\).■

\(\text{Or}_2\) will be represented by the group denoted by \(7,1,11,1(0)\) in [1]. The fundamental group \(G \sim \mathbb{Z}_2 \oplus \mathbb{Z}_2\) will be a free product of two Coxeter groups: \(G = \langle f_1 \rangle \ast \langle f_2 \rangle\). Here

\[
(2.13) \quad f_1 : (\varphi, \vartheta, y) \mapsto (\varphi + \pi, -\vartheta, -y) \in S^2 \mathbb{R}_1
\]

\[
(2.14) \quad f_2 : (\varphi, \vartheta, y) \mapsto (\varphi + \pi, -\vartheta, -y + 1) \in S^2 \mathbb{R}_1
\]

are two involutive generators of \(G\) whose elements are

\[
(2.14) \quad 1, \tau := f_1 f_2, \tau^{-1} := f_2 f_1, \ldots, \tau^n, \tau^{-n}, \ldots, n = 0, 1, \ldots (\sim \mathbb{Z})
\]

\[
\tau^k f_1 = f_1 \tau^{-k}, \ldots, \tau^{-k} f_2 = f_2 \tau^k, \ldots, k = 0, 1, \ldots
\]

By other words: \(G\) is an infinite dihedral group, or \(G\) is a free Coxeter group of 2 generators (see Fig.1 for 2 geometric presentations of \(\text{Or}_2\)).
A systematic enumeration of $S^2 \times \mathbb{R}$ space forms

In Table 1 there are listed the finite isometry groups $A$ of $S^2$ in different notations, from which we prefer the 2-orbifold signatures of Macbeath and Conway, equivalent to each other. Here the factor surface $S^2/A$ is characterized by the $A$-orbits of $S^2$. Any fundamental domain $\mathcal{F}_A$ with a side pairing - as usual - provides us a more visual picture (Fig. 3).

E.g. the group

$$1q - (\pm, 0; [q,q]; \{\}) \quad q \geq 1 \rightarrow q, q$$

is generated by

$$r : (\varphi, \theta) \mapsto (\varphi + \frac{2\pi}{q}, \theta)$$

a a $q$-fold rotation of $S^2$.

A 2-gon (digon) with $\frac{2\pi}{q}$ angles at the opposite poles and with pairing the (may be bent) sides by $r$, will topologically be an orientable (+) surface of genus 0 (a sphere), where the two opposite $q$-fold rotational centres are distinguished (as two cone points) by $\frac{2\pi}{q}$ angular neighbourhood of $S^2$ at each pole (Fig. 3).

<table>
<thead>
<tr>
<th>Macbeath signature</th>
<th>H. Weyl</th>
<th>Schoen-</th>
<th>Coxeter-</th>
<th>Conway</th>
</tr>
</thead>
<tbody>
<tr>
<td>1q</td>
<td>(+, 0; [q,q]; {}) $q \geq 1$</td>
<td>$C_q$</td>
<td>$C_q$</td>
<td>{q} *</td>
</tr>
<tr>
<td>2q</td>
<td>(+, 0; 1; {(q,q)}) $q \geq 2$</td>
<td>$D_q C_q$</td>
<td>$C_q$</td>
<td>{q} *</td>
</tr>
<tr>
<td>3q</td>
<td>(+, 0; 2, q; {}) $q \geq 2$</td>
<td>$D_q$</td>
<td>$D_q$</td>
<td>{2q} *</td>
</tr>
<tr>
<td>4q</td>
<td>(+, 0; {(2, 2, q)}) $q \geq 3$</td>
<td>$D_{2q} D_q$</td>
<td>$D_{2q} h$</td>
<td>{2q} *</td>
</tr>
<tr>
<td>5q</td>
<td>(+, 0; q; {(1)}) $q \geq 1$</td>
<td>$C_{2q} C_q$</td>
<td>$C_{2q} h$</td>
<td>{2q} *</td>
</tr>
<tr>
<td>5e</td>
<td>(+, 0; q; {(1)}) $q \geq 2$</td>
<td>$C_q \times I$</td>
<td>$C_{2q} h$</td>
<td>{2q} *</td>
</tr>
<tr>
<td>6q</td>
<td>(+, 0; 2; {(q)}) $q \geq 3$</td>
<td>$D_q \times I$</td>
<td>$D_{2q} h$</td>
<td>{2q} *</td>
</tr>
<tr>
<td>7qe</td>
<td>(+, 0; 2; {(q)}) $q \geq 2$</td>
<td>$D_{2q} D_q$</td>
<td>$D_{2q} h$</td>
<td>{2q} *</td>
</tr>
<tr>
<td>11</td>
<td>(+, 0; {(2, 3, 3)})</td>
<td>$A_4$</td>
<td>$T$</td>
<td>3, 3 *</td>
</tr>
<tr>
<td>12</td>
<td>(+, 0; {(2, 3, 4)})</td>
<td>$S_4 \times I$</td>
<td>$O_h$</td>
<td>3, 4 *</td>
</tr>
<tr>
<td>13</td>
<td>(+, 0; {(2, 3, 5)})</td>
<td>$A_5 \times I$</td>
<td>$I_h$</td>
<td>3, 5 *</td>
</tr>
</tbody>
</table>

Table 1.

To form appropriate $S^2 \times \mathbb{R}$ space form group $G$ from $q, q$ above, we choose first a point group $G_0$ by (2,3) then the translational parts by (2,4), so that the
Figure 2. Symbolic picture for diffeomorphism equivariance by a skew transform $S$. $s = S^{-1}rS$. Here $p$ denotes the angle $\alpha$.

Figure 3. Spherical groups for $S^2 \times \mathbb{R}$ space forms
(a) $1q - q, q$ for Or1
(b) $5q - q$ for No1 and No2
(c) $7q - q$ for No1, No2, and Or2, respectively.
group $G$ by (2.2) shall be fixed point free. We recall from [1] the three types of $S^2 \times \mathbb{R}$ point groups derived from any isometry group $A$ of $S^2$:

$$T_{\text{type I}}: G_0 = A \times \mathbb{R}, \quad T_{\text{type II}}: G_0 = A \times T_{\mathbb{R}}$$

$$T_{\text{type III}}: G_0 = A' B := \{ B \times \mathbb{R} \} \cup \{ (A \setminus B) \times T_{\mathbb{R}} \}$$

where $B$ is a subgroup in $A$ of index two.

$T_{\text{type I}}$: $(q, q) \times \mathbb{R}$ from (3.1) has the presentation

$$(g_1 - g_0^q) g_1 \in S^2_2$$

with one generator $g_1 := r \times \mathbb{R}$ and relation $g_1^q = 1$. The possible translational part $k_1$ in $(g_1, k_1)$ satisfies, by (2.4), the so called Frobenius congruence

$$k_1 q \equiv 0 \pmod{1}$$

implying $k_1 \equiv 0$ or $k_1 \equiv \frac{k}{q} \quad k = 1, \ldots, q - 1$.

The first solution leads to fixed point free group if $q = 1$, the second ones make this if $(k, q) = 1$, just as we have described in Sect.2 (in Rem. 4, formula (2.11), Prop.2.3).

$T_{\text{type II}}$: $(q, q) \times T_{\mathbb{R}}$ from (3.1) has the presentation

$$(g_1, g_2 - g_1^q, g_2, g_1^{-1} g_2 g_1, g_2), \quad g_1 \in S^2_2, \quad g_2 \in \mathbb{R}$$

for $g_1 := g \times \mathbb{R}$ and $g_2 : (\varphi, \theta, y) \mapsto (\varphi, \theta, -y)$. The translational parts $k_1$ and $k_2$ in $(g_1, k_1)$ and $(g_2, k_2)$ satisfy the Frobenius congruences

$$k_1 q \equiv 0, \quad k_2 q \equiv 0, \quad k_1 k_2 \equiv 0 \pmod{1}.$$ 

Now we have only to emphasize that for any $k_2 \in \mathbb{R}$

$$(g_2, k_2) : (\varphi, \theta, y) \mapsto (\varphi, \theta, -y + k_2) \in \mathbb{R}_1$$

is a reflection in the $S^2 \times \{ \frac{1}{q} k_2 \}$ level with fixed points.

Thus we do not obtain any space form group in the $T_{\text{type II}}$.

$T_{\text{type III}}$: $(q, q)/(q, \frac{2\pi}{q})$ with $2 \leq q$ even yields a presentation

$$(g_1 - g_0^q), \quad g_1 : (\varphi, \theta, y) \mapsto (\varphi + \frac{2\pi}{q}, \theta, -y) \in S^2_2 \mathbb{R}_1$$

Any transform $(g_1, k_1), \quad k_1 \in \mathbb{R}$ for any even $q \geq 2$, has fixed points $(., \frac{\pi}{q}, \frac{2\pi}{q})$, $(., -\frac{\pi}{q}, \frac{2\pi}{q})$ over the poles of $S^2$, yielding no space form in this type.

The next important isometry group series of $S^2$ (Table 1) is

$$7q - (-1; [q]; \{\}) \quad q \geq 1 - q^2.$$ 

Here every 2-orbifold $(S^2/A)$ is nonorientable $(-)$ surface with genus 1 (i.e. a projective plane, or i.e., the sphere with one cross cap $\odot$) with a rotation centre of order $q$ (cone point with angular neighborhood $\frac{2\pi}{q}$), in Fig.3 we have pictured
its symbolic fundamental domain $\mathcal{F}_{q\otimes}$ with its side pairing. This provides the generator

$$z : (\varphi, \vartheta) \mapsto (\varphi + \frac{\pi}{q}, -\vartheta) \in S^3_0$$

a rotary reflection of $S^2$.

The possible point groups as follow:

- Type I: $q^\otimes \times 1_R$ has the presentation

$$(g_1 - g_1^{2q}), \ g_1 \in S^3_0 : g_1 : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{\pi}{q}, -\vartheta, y),$$

$k_1$ in $(g_1, k_1)$ satisfies the Frobenius congruence

$$k_1 2q \equiv 0 \pmod{1} : k_1 \equiv 0; \ k_1 \equiv \frac{1}{2}; \ k_1 \equiv k, \ k = 1, 2, \ldots, q - 1.$$

The diffeomorphism class No1 of nonorientable $S^2 \times R$ space forms will be represented by 7.1.1.1(0) from [1], i.e. in case $q = 1, \ k_1 = 0, \ z := g_1$ (Fig. 4). The fundamental domain $\mathcal{F}_G$ of this $G$ is a “half shell” with unusual face pairing which provides the presentation (by unusual “edges”)

$$G = (z, \tau - z^2, \tau z^{-1}) \sim Z_2 \times Z.$$
The second diffeomorphism class No2 of nonorientable $S^2 \times \mathbb{R}$ space forms will be represented by $7.1.1.2(\frac{1}{2})$ from [1], i.e. in case $q = 1$, $k_1 = \frac{1}{2}$ (Fig.4). The fundamental domain $\mathcal{F}_G$ is again a “half shell” with another face pairing with presentation

\[ G = (z, \vec{z}, \tau - z\vec{z}\tau^{-1}, \bar{z}z\tau^{-1}, x\bar{z}^{-1}, \bar{x}^{-1}z) \sim \mathbb{Z}, \]

since $\bar{z} = z$, $\tau = z\bar{z}$ are consequences. Other $q > 1$ leads to fixed points over the poles of $S^2$ in both above cases $k_1 = 0$ or $k_1 = \frac{1}{2}$.

The third case in (3.12) yields fixed point free group iff the g.c.d $(k, q) = 1$. Then the generator of the group $G$

\[ z := (g_1, k_1) : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{\pi}{q}, -\vartheta, y + \frac{k}{2q}) \]

leads to cases: i, $q$ odd, $k$ even ii, $q$ odd, $k$ odd iii, $q$ even.

i, $1 < q$ odd, $k = 2u_1, 1 \leq u \in \mathbb{N}$. Consider the element

\[ w := z^{2t^2} : (\varphi, \vartheta, y) \mapsto (\varphi + \pi, -\vartheta, y) \in S^2_2 \]

which is just the antipodal map of $S^2$, an orientation reversing involution, i.e. $ww = 1$. The following element $\vec{z}$ - with $t, v$ odd - will be

\[ \vec{z} := z^v \tau^{-t} : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{v\pi}{q}, -\vartheta, y + \frac{1}{q}) \in S^2 \mathbb{R}_2, \]

here $2uv - 2tq = 2$ , i.e. $uv - tq = 1$, because of g.c.d. $(u, q) = 1$ can be chosen. This provides a minimal (non zero) translational part, uniquely, since different $v_1, v_2$ (mod $q$) could not serve this translational part. Then

\[ G = \langle w \rangle \times \langle \vec{w} \rangle \sim \mathbb{Z}_2 \times \mathbb{Z}, \]

and the skew transform $S$ by (2.8) with $\alpha = \frac{i\varphi + \vartheta}{q}$, $a = \frac{1}{q}$ shows that $S^2 \times \mathbb{R}/G$ belongs to the diffeomorphism class No1 by (3.13). To this, following Prop.2.2, we can check with $z$ in (3.13) that

\[ w = S^{-1}zS, \quad w\vec{z} = S^{-1}\tau S \]

hold, indeed.

In cases ii, and iii, $z$ in (3.15) does not produce an involutive element of $G$. With appropriate integers $t, v$ odd we take

\[ \vec{z} := z^v \tau^{-t} : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{v\pi}{q}, -\vartheta, y + \frac{1}{2q}) \in S^2 \mathbb{R}_2 \]

where $kv - 2qt = 1$ since g.c.d $(k, 2q) = 1$.

This $\vec{z}$ provides a minimal (non zero) translational part, uniquely, since different $v_1, v_2$ (mod $q$) could not serve this translational part (we may apply also the similarity (2.10)). Then

\[ G = \langle \vec{z} \rangle \sim \mathbb{Z} \]

leads to the diffeomorphism class No2 by (3.14), again by the skew transform $S$ in (2.8).


Type II: \( q \otimes \mathbb{R} \) leads to fixed points analogously as before. We do not obtain any \( S^2 \times \mathbb{R} \) space form.

Type III: \((q, q') \) has the presentation

\[
(g_1 - g_2^n), \quad g_1 : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{\pi}{q}, -\vartheta, -y) \in S^2_{\mathbb{R}}.
\]

The translational part \( k_1 \) in \((g_1, k_1)\) satisfies by (2.4)

\[
k_1 \mapsto (-k_1) + k_1 = 0 \mapsto k_1 \ldots \mapsto 0.
\]

Thus, by choosing the similarity \( g : (\varphi, \vartheta, y) \mapsto (\varphi, \vartheta, y + \frac{1}{q}k_1) \) (as translation), we get equivariance to case \( k_1 = 0 \). We use the notation \( f := g_1 \) for this involutive transform in case \( q = 1 \), which is the product of the antipodal map of \( S^2 \) and a reflection in \( S^2 \times \{0\} \). Else \((q > 1)\) we obtain fixed points over the poles of \( S^2 \).

Thus we get the promised representative \( S^2 \times \mathbb{R} / G \) for the second diffeomorphism class \( \text{Or} 2 \) of orientable \( S^2 \times \mathbb{R} \) space forms in Fig.1 with

\[
G := (f, \tau - f^2, f \tau f \tau)
\]

by half shell \( \mathcal{F}_2 \). Or equivalently \( G := (f_1, f_2 - f_2^2, f_2^3) \) holds by a shell \( \mathcal{F}_1 \), if \( f_1 := f \), and \( f_2 := \tau : (\varphi, \vartheta, y) \mapsto (\varphi + \pi, -\vartheta, -y + 1) \) for the free Coxeter group \( \langle f_1 \rangle \times \langle f_2 \rangle \).

This was summarized at the end of Sect.2.

4. THE OTHER SIMILARITY CLASSES OF \( S^2 \times \mathbb{R} \) SPACE FORMS

After the discussions detailed before, we treat the remaining cases more sketchily.

From the finite isometry groups of \( S^2 \) only (Fig.3)

\[
5q - (+, 0; [q]; \{1\}), \quad 1 \leq q \in \mathbb{N} - q^*
\]

provide \( S^2 \times \mathbb{R} \) space forms. From the other spherical groups in Table 1 each leads to fixed points as in [1]. So the classification of orientable space forms has already been complete.

Any group \( A \) in (4.1) are generated by a reflection \( g_1 \) in an equatorial circle of \( S^2 \) and by a \( q \)-fold rotation \( g_2 \) about its poles. The fundamental domain \( \mathcal{F}_A \) in Fig.3 shows also the \( S^2 \)-orbifold with one boundary component in \( \{1\} \) or the empty sign after \(*\), i.e. without non trivial dihedral corner; moreover, if \( q > 1 \), one \( q \)-fold rotation centre (one point of angular neighborhood \( \frac{2\pi}{q} \)) in \([q]\) or \( q \) before \(*\), respectively.

Again, we consider the possible 3 types of point groups \( G_0 \) and the corresponding \( S^2 \times \mathbb{R} \) space groups \( G \) without any fixed point.

Type I: \( q^* \times 1_{\mathbb{R}} \) has the presentation

\[
(g_1, g_2 - g_1^2, g_2^2, g_2^{-1} g_1 g_2 g_1),
\]

\[
g_1 : (\varphi, \vartheta, y) \mapsto (\varphi, -\vartheta, y); \quad g_2 : (\varphi, \vartheta, y) \mapsto (\varphi + \frac{2\pi}{q}, \vartheta, y).
\]
The translational parts \( k_1 \) in \((g_1, k_1)\) and \( k_2 \) in \((g_2, k_2)\) satisfy the Frobenius congruences

\[
(4.3) \quad k_1 2 \equiv 0, \quad k_2 q \equiv 0, \quad -k_2 + k_1 + k_2 + k_1 = 2k_1 \equiv 0 \quad (\text{mod } 1),
\]

Only \((k_1, k_2) = (k, \frac{k}{q})\) provides nonorientable \(S^2 \times \mathbb{R}\) space forms. Then

\[
(4.4) \quad a := (g_1, \frac{k}{2}) : (\varphi, \theta, y) \mapsto (\varphi - \theta, y + \frac{1}{2})
\]

\[
s := (g_2, \frac{k}{q}) : (\varphi, \theta, y) \mapsto (\varphi + \frac{2\pi}{q}, \theta, y + \frac{k}{q})
\]

with \(\gcd(k, q) = 1\) and \(k = 1, \ldots, \lfloor \frac{q}{2} \rfloor\) for l.p of \(\frac{q}{2}\), generate our group \(G\). We discuss the two cases: i. \(q = 2u\) even and ii. \(q\) odd.

i. If \(q = 2u\) is even, we take

\[
(4.5) \quad w := a^u \tau^{-t} : (\varphi, \theta, y) \mapsto (\varphi + \frac{2\pi u}{2u}, -\theta, y) \in S^2_a
\]

by \(\frac{1}{2} + \frac{k}{2u} - t = 0\), i.e. \(2u(1 - 2t) + 2k v = 0\) holds with \(v = u = \frac{q}{2}\), \(k = 2t - 1\), \(ww = 1\), thus \(w\) is the involutive antipodal map. The transform

\[
(4.6) \quad s : s^u \tau^{-r} : (\varphi, \theta, y) \mapsto (\varphi + \frac{2\pi s}{u}, \theta, y + \frac{1}{q}) \in S^2_a \mathbb{R}_2,
\]

where

\[
\frac{ks}{2u} - r = \frac{1}{2u}, \quad \text{i.e. } ks - 2ur = 1 \text{ since } \gcd(k, 2u) = 1,
\]

is just the unique screw motion in \(G\) with minimal non-zero translational part. We see, that

\[
(4.7) \quad G = \langle w \rangle \times \langle s \rangle \sim Z_2 \times Z
\]

serves an \(S^2 \times \mathbb{R}\) space form diffeomorphic to \(\mathbb{N} \mathbb{O} 1\) by a skew transform \(S\) by (2.8) as earlier. To this

\[
(4.8) \quad a := w^s \tau^{-t} s^{-u} = w^s (\varphi - ku) = ws^s,
\]

\[
s = s^u,
\]

by (4.5) and (4.6) are satisfactory equations, according to (4.4).

ii. If \(q\) is odd, then we can take with integer \(s\) and \(t\)

\[
(4.9) \quad \overline{z} : a^s \tau^{-t} : (\varphi, \theta, y) \mapsto (\varphi + \frac{2\pi s}{q}, -\theta, y + \frac{1}{2q}) \in S^2_d \mathbb{R}_2
\]

by

\[
\frac{1}{2} + \frac{s k}{q} - t = \frac{1}{2q}, \quad \text{i.e. } 2sk + (1 - 2t)q = 1 \text{ by } \gcd(2k, q) = 1,
\]

as a unique generator. Moreover, \(G\) does not have any involutive element now. Namely, we can express the generators in (4.4) by \(\overline{z}\):

\[
(4.10) \quad a = \overline{z}^a, \quad s = \overline{z}^{ak}
\]
This proves \( G = (\mathbb{I}) \sim \mathbb{Z} \), and a skew transform \( S \) by (2.8) shows the
diffeomorphic equivariance to \( G \) in (3.20-21) in the class \( \text{No2} \).

Type II: \( q^* \times T_x \mathbb{R} \) leads to fixed points as before.

Type III: We have three possibilities as in [1]:

\[ 3.11 \quad \text{III.} \, a (q^*)' (q, q), \quad \text{III.} \, b (q^*)' (\frac{q}{2}^*), \quad \text{III.} \, c (q^*)' (\frac{q}{2} \otimes), \]

where \( q \) is even in the last two cases. Again, we have fixed points at each space

\[ 3.12 \quad \mathbb{Z} : (\varphi, \vartheta, y) \mapsto (\varphi + \alpha, -\vartheta, y + \alpha) \in S^2 \mathbb{R}_2 \]

for \( G \) with irrational \( \frac{\alpha}{2\pi} \in (0, \frac{1}{2}) \); \( 0 < \alpha \in \mathbb{R} \).

This is as to in the orientable class \( \text{Or1} \) with the screw motion \( s \) in (2.6).

2, In the orientable case \( f \in S^2 \mathbb{R}_2 \) by (3.13) is the only (even similarity) type of

involutive transforms without fixed points. Combining this \( f \) with a screw motion

\[ 3.13 \quad \text{fsfs} : (\varphi, \vartheta, y) \mapsto (\varphi + 2\pi + 2\alpha, \vartheta, y) \]

has fixed points over the poles. Thus we see that the diffeomorphism class \( \text{Or2} \) has

exactly one similarity class of \( S^2 \times \mathbb{R} \) space forms, and we do not have any more.

3, In the nonorientable case the antipodal map \( z \in S^2 \mathbb{R} \) in (3.13) is the only type of

involutive transforms without fixed points. Combining this \( z \) with any screw motion \( s \) in (2.6)

\[ 3.14 \quad zsz : (\varphi, \vartheta, y) \mapsto (\varphi + 2\pi + \alpha, \vartheta, y + \alpha), \quad zsz = s \]

show that our diffeomorphism class \( \text{No1} \) contains also uncountably many similarity

classes, and we do not have any more.

At the end we summarize our results in

**Theorem 4.1** There are exactly 2 orientable: \( \text{Or1}(\mathbb{Z}) \) and \( \text{Or2}(\mathbb{Z} \otimes \mathbb{Z}) \), resp. 2

nonorientable diffeomorphism classes: \( \text{No1}(\mathbb{Z} \times \mathbb{Z}) \) and \( \text{No2}(\mathbb{Z}) \) of \( S^2 \times \mathbb{R} \) space

forms, containing the similarity equivariance classes as follows in Table 2. In the
diffeomorphism class \( \text{Or2} \) there are exactly one similarity type, also in the general

sense if we allow infinite point groups for the fundamental groups. The other 3
diffeomorphism classes contain similarity classes in infinite series for finite point

groups, or uncountably many similarity classes for infinite point groups.\]
Table 2 Classes of $S^2 \times \mathbb{R}$ space forms

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Conditions</th>
<th>Diffeomorphism class</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,1,1,1(0)</td>
<td>representative (Fig.1)</td>
<td>$O(1)(\mathbb{Z})$</td>
</tr>
<tr>
<td>1q.1.2(h/2)</td>
<td>$(k,q) = 1, \ 1 \leq k \leq \lfloor \frac{h}{2} \rfloor$</td>
<td>$O(1)$</td>
</tr>
<tr>
<td>7.1,1.1(0)</td>
<td>representative (Fig.4)</td>
<td>$No(1)(\mathbb{Z} \times \mathbb{Z})$</td>
</tr>
<tr>
<td>7.1,1.2(1/2)</td>
<td>repr. (Fig.4)</td>
<td>$No(2)(\mathbb{Z})$</td>
</tr>
<tr>
<td>7qo.1.3(h/2q)</td>
<td>$2 \leq q \text{ odd}, (k,q) = 1, \text{ even } k &lt; q$</td>
<td>$No1$</td>
</tr>
<tr>
<td>7qo.1.3(h/2q)</td>
<td>$2 \leq q \text{ odd}, (k,q) = 1, \text{ odd } k &lt; q$</td>
<td>$No2$</td>
</tr>
<tr>
<td>7qe.1.3(h/2q)</td>
<td>$2 \leq q \text{ even}, (k,q) = 1, k &lt; q$</td>
<td>$No2$</td>
</tr>
<tr>
<td>7.1.III.1(0)</td>
<td>repr. (Fig.1)</td>
<td>$O(2)(\mathbb{Z} \otimes \mathbb{Z})$</td>
</tr>
<tr>
<td>5qe.1.4(h/2, k/2)</td>
<td>$2 \leq q \text{ even}, (k,q) = 1, 1 \leq k \leq \lfloor \frac{h}{2} \rfloor$</td>
<td>$No1$</td>
</tr>
<tr>
<td>5qe.1.4(h/2, k/2)</td>
<td>$1 \leq q \text{ odd}, (k,q) = 1, 1 \leq k \leq \lfloor \frac{h}{2} \rfloor$</td>
<td>$No2$</td>
</tr>
</tbody>
</table>

References


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