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CONSTRUCTIONS OF THE MAXIMAL EXCEPTIONAL GRAPHS WITH LARGEST DEGREE LESS THAN 28

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Abstract. A graph is said to be exceptional if it is connected, has least eigenvalue greater than or equal to -2, and is not a generalized line graph. Such graphs are known to be representable in the exceptional root system E_8. The 473 maximal exceptional graphs have been found by computer, and the 467 with maximal degree 28 have been characterized. Here we construct the remaining six maximal exceptional graphs by means of representations in E_8.

1. Introduction

An *exceptional* graph is a connected graph with least eigenvalue greater than or equal to -2 which is not a generalized line graph. Exceptional graphs first appeared in the context of spectral characterizations of certain classes of line graphs by A. J. Hoffman and others in the 1960s (see, for example, [12, pp. 12-14]). The key paper [5] introduced root systems as a means of investigating graphs with least eigenvalue -2; in particular it was shown by this technique that an exceptional graph has at most 36 vertices and each vertex has degree at most 28. The regular exceptional graphs, 187 in number, were found in [2, 3], but the problem of finding a suitable description of all the exceptional graphs remained open. Generalized line graphs have been studied in [9] and [14]. Much information on these topics can be found in the monographs [1, 6, 8] and in the expository paper [4]. We described in [11] the 473 exceptional graphs which are maximal in the sense that every exceptional graph is an induced subgraph of (at least) one such graph. These maximal exceptional graphs were initially found by computer using the star complement technique, and the 467 with maximal degree 28 were characterized in [11]. Here we construct the six maximal exceptional graphs with maximal degree less than 28.

It is well known that an exceptional graph G is representable in the root system E_8 (see [6, Chapter 3] or [1, Chapter 3]). This means that if G has A as a $(0, 1)$-adjacency matrix then $I + \frac{1}{2}A$ is the Gram matrix of a set of normalized vectors in E_8; explicitly, if $\{\mathbf{e}_1, \ldots, \mathbf{e}_8\}$ is an orthonormal basis for $I\!R^8$ then $8I + 4A$ is the Gram matrix of a subset of the following set of 240 vectors (cf. [2]):

type a: 28 vectors of the form $\mathbf{a}_{ij} = 2\mathbf{e}_i + 2\mathbf{e}_j$; $i, j = 1, \ldots, 8$, $i < j$;

type a′: 28 vectors opposite to those of type a;

type b: 28 vectors of the form $\mathbf{b}_{ij} = -2\mathbf{e}_i - 2\mathbf{e}_j + \sum_{k=1}^{8} \mathbf{e}_k$;

type b′: 28 vectors opposite to those of type b;

type c: 56 vectors of the form $\mathbf{c}_{ij} = 2\mathbf{e}_i - 2\mathbf{e}_j$; $i, j = 1, \ldots, 8$, $i \neq j$;

type d: 70 vectors of the form $\mathbf{d}_{ijkl} = 2\mathbf{e}_i + 2\mathbf{e}_j + 2\mathbf{e}_k + 2\mathbf{e}_l - \sum_{s=1}^{8} \mathbf{e}_s$ with distinct $i, j, k, l \in \{1, \ldots, 8\}$;

type e: 2 vectors $\mathbf{e}$ and $-\mathbf{e}$, where $\mathbf{e} = \sum_{i=1}^{8} \mathbf{e}_i$.

These 240 vectors determine 120 lines at 60° or 90°. Let Γ be the graph which has these lines as vertices, with two vertices adjacent if and only if the corresponding lines are orthogonal. (This is the complement of the graph denoted by $O^-(8, 2)$ in [2].) In the notation of [7], the automorphism group of Γ has the form $O^+(8, 2).2$. It is transitive on vertices, and the stabilizer of a vertex v is edge-transitive on the subgraph induced by the neighbours of v (cf. [7, p.85]).

Here a *representation* of the exceptional graph G is a subset $\mathcal{R}(G)$ of E_8 whose Gram matrix is a scalar multiple of $8I + 4A$, where A is the adjacency matrix of G. In view of the transitivity of $\mathrm{Aut}(\Gamma)$, we assume that $\mathbf{e}$ represents a vertex of maximal degree, and in this case we call $\mathcal{R}(G)$ a *standard* representation. Note that then no vector of type a' or b' features in $\mathcal{R}(G)$; moreover a second standard representation is given by $\phi(\mathcal{R}(G))$ where: $\phi(\mathbf{e}) = \mathbf{e}$, $\phi(\mathbf{a}_{ij}) = \mathbf{b}_{ij}$, $\phi(\mathbf{b}_{ij}) = \mathbf{a}_{ij}$, $\phi(\mathbf{c}_{ij}) = \mathbf{c}_{ji}$ $(= -\mathbf{c}_{ij})$, $\phi(\mathbf{d}_{ijkl}) = \mathbf{d}_{\overline{ijkl}}$ $(= -\mathbf{d}_{ijkl})$. (Accordingly we may assume if necessary that the number of vectors of type a in $\mathcal{R}(G)$ does not exceed the number of vectors of type b.)

For future reference, we note that the line graph $L(K_8)$ can be represented by all vectors of type a, or by all vectors of type b: in both cases the Gram matrix of the vectors is $8I + 4A$, where A is an adjacency matrix of $L(K_8)$. Replacing some vectors of type a by the corresponding vectors of type b (or vice-versa) is equivalent to switching with respect to those vectors in the sense of Seidel [6, p. 59]. We say that two graphs are *switching-equivalent* if one can be obtained from the other by switching.

The 473 maximal graphs are denoted in [11] by $M001$, $M002, \ldots, M473$. The graphs are ordered by the number of vertices, the number of edges ($M003$ excepted), and then by vertex degrees, with all these invariants in non-decreasing order.

The distribution over number of vertices of the maximal exceptional graphs is as follows:

Number of vertices	22	28	29	30	31	32	33	34	36
Number of graphs	1	1	432	25	7	3	1	2	1

For any graph G, the *cone* over G is the graph $K_1 \nabla G$ obtained from G by adding a new vertex adjacent to all vertices of G. The maximal exceptional graphs with 29 vertices are all cones except for two, namely $M417$ and $M428$. The graphs $M001$ and $M002$ are not cones while those on more than 29 vertices cannot be since the maximal vertex degree of graphs representable in E_8 is 28. The graph $M001$ is mentioned in [15] as a non-regular graph with three distinct eigenvalues. The graph $M007$ is the cone over $L(K_8)$ while $M004$, $M005$ and $M006$ are cones over the Chang graphs [12, Example 1.1.2]. The graph $M003$ is the double cone $K_2 \nabla G$ where G is the Schläfli graph [6, p. 32].

All of the maximal exceptional graphs are non-regular, and in [11] they are classified into three types: (a) those which are 29-vertex cones, (b) those which have a vertex of degree 28 but have more than 29 vertices, (c) those in which each vertex has degree less than 28. In the notation of [11]. the graphs of type (c) are $M001, M002, M417, M428, M437$ and $M462$, with vertex degrees as follows:

- $M001$: 16^{14}, 7^8 (the vertices of degree 16 induce the cocktail-party graph $\overline{7K_2}$, while those of degree 7 form a coclique);
- $M002$: 22^7, 16^{14}, 10^7 (the vertices of degree 10 form a coclique);
- $M417$: 26^1, 24^2, 18^{16}, 12^8, 10^2;
- $M428$: 26^2, 22^1, 18^{16}, 14^6, 10^4;
- $M437$: 26^2, 24^1, 20^8, 17^8, 16^1, 14^2, 13^4, 11^4;
- $M462$: 26^3, 22^4, 19^8, 16^4, 15^6, 12^6.

These graphs will be defined explicitly in the next section.

Let $G(P)$ denote the cone over the graph obtained from $L(K_8)$ by switching with respect to the edge-set $E(P)$, where P is a spanning subgraph of K_8. (Thus for each edge ij of P the a type vector $2\mathbf{e}_i + 2\mathbf{e}_j$ is replaced by the corresponding b type vector $\mathbf{e} - 2\mathbf{e}_i - 2\mathbf{e}_j$.) We define properties (I) and (II) of P as follows:

(I) P has a 4-clique and a 4-coclique on disjoint sets of vertices,
(II) P has six vertices adjacent to a seventh and non-adjacent to the eighth.

These configurations are called *dissections* of P of type I or II. A dissection of type I yields a partition of the vertex set into two subsets of cardinality 4, while a dissection of type II yields a partition into subsets of cardinalities 6 and 2.

The maximal exceptional graphs of type (a) and (b) are characterized in [11] by the following theorem.

Theorem 1. *Let G be an exceptional graph with $29+k$ vertices ($k \geq 0$), and suppose that G has a vertex u of degree 28. Let Y be the set of vertices not adjacent to u. Then G is a maximal exceptional graph if and only if $G-Y$ is isomorphic to a cone $G(P)$ in which P has exactly k dissections.*

In this report we construct the maximal exceptional graphs of type (c) using a generalization of the spanning subgraph P.

2. Graphical representations of exceptional graphs

A standard representation $\mathcal{R}(G)$ in E_8 of an exceptional graph G can be visualized in terms of its *root graph*: this is a graph on the *points* $1,2,3,4,5,6,7,8$ with red edges ij corresponding to $\mathbf{a}_{ij}$ and blue edges ij corresponding to $\mathbf{b}_{ij}$, together with directed *arcs* $\vec{ij}$ corresponding to $\mathbf{c}_{ij}$, *curves* $ijkl$ corresponding to $\mathbf{d}_{ijkl}$ and the *ideal element* $\{1,2,3,4,5,6,7,8\}$ corresponding to $\mathbf{e}$. (Note that the points i,j,k,l of a curve are the co-ordinates of the positive entries of $\mathbf{d}_{ijkl}$.) We say that two such objects are *incompatible* if two corresponding vectors are incompatible, that is, if their scalar product is -4. Accordingly we have:

a) the ideal element is compatible with all others;

b) distinct edges are always compatible; they give rise to a switched line graph: incident lines of the same colour give rise to adjacent vertices;

c) two arcs are compatible if and only if they do not induce an (oriented) path of length two;

d) two curves are compatible if and only if they intersect in at least two points (or, equivalently, if and only if the Hamming distance between the corresponding 8-tuples is at most four); two curves give rise to adjacent (non-adjacent) vertices if and only if they have three (resp. two) common points;

e) a red (blue) edge is incompatible with an arc if and only if it is non-parallel with it, but incident to its tail (resp. head); adjacent vertices arise if and only if the red (blue) edge and the arc have only the head (resp. tail) of the arc as a common element;

f) a red (blue) edge is in incompatible with a curve if and only if it disjoint from it (resp. contains its two points); adjacent vertices arise if and only if the red (blue) edge and the curve meet in two (resp. zero) points;

g) an arc and a curve are incompatible if and only if only the tail of the arc belongs to the curve; adjacent vertices arise if and only if the head, but not the tail, of the arc is on the curve;

In what follows we shall say that two graphical objects are adjacent whenever they correspond to vectors representing adjacent vertices of G. For a vector $\mathbf{v} \in \mathcal{R}(G)$ we shall write $\deg(\mathbf{v})$ for the degree of the corresponding vertex in G.

Note that the graph G is a maximal exceptional graph if and only if it has a standard representation whose root graph cannot be extended by further red or blue edges, arcs or curves. The maximal exceptional graphs of type (c), found in [11] by a computer search, have standard representations for which the objects of the root graphs, in addition to the ideal element, are as follows:

1^o $M001$ (22 vertices): [red edges] 12 13 14 23 24 34 15 26 37 48; [blue edges] 56 57 58 67 68 78; [arcs] $\vec{15}$ $\vec{26}$ $\vec{37}$ $\vec{48}$; [curves] 1234.

2^o $M002$ (28 vertices): [red edges] 12 13 17 23 27 37 14 25 36 78 18 28 38; [blue edges] 45 46 48 56 58 68 47 57 67; [arcs] $\vec{14}$ $\vec{25}$ $\vec{36}$; [curves] 1237 1238.

3^o $M417$ (29 vertices): [red edges] 12 13 14 15 16 17 18 34 35 36 47 48; [blue edges] 25 26 27 28 37 38 45 46 56 57 58 67 68 78; [arcs] $\vec{12}$; [curves] 1234.

4^o $M428$: (29 vertices): [red edges] 12 13 14 15 16 17 18 34 35 36 37 48; [blue edges] 25 26 27 28 38 45 46 47 56 57 58 67 68 78; [arcs] $\vec{12}$; [curves] 1234.

5^o $M437$: (30 vertices): [red edges] 12 15 16 17 18 25 26 27 28 56; [blue edges] 13 24 34 35 36 37 38 45 46 47 48 57 58 67 68 78; [arcs] $\vec{13}$ $\vec{24}$ [curves] 1256.

6^o $M462$: (31 vertices): [red edges] 12 15 16 17 18 25 26 27 28 56 67; [blue edges] 13 24 34 35 36 37 38 45 46 47 48 57 58 68 78; [arcs] $\vec{13}$ $\vec{24}$; [curves] 1256 1267.

These standard representations are not unique and we shall encounter others in the sequel. In each case the necessary isomorphism can be established either by a simple computer program or by identifying an appropriate star complement (cf. [11, Section 3.2]).

Henceforth we consider a standard representation $\mathcal{R}(G)$ of a maximal exceptional graph G with maximal degree less than 28. From the proof of [11, Theorem 3.6] we know that $\mathcal{R}(G)$ contains vectors $\mathbf{v}, \mathbf{w}$ such that $\mathbf{e}, \mathbf{v}, \mathbf{w}$ are pairwise orthogonal. Since the stabilizer of the line $\langle \mathbf{e} \rangle$ in $\mathrm{Aut}(\Gamma)$ is edge-transitive on the subgraph induced by the neighbours of $\langle \mathbf{e} \rangle$, we may assume that $\mathbf{v}, \mathbf{w}$ are vectors of type c, equivalently that the corresponding root graph has two non-adjacent arcs. Let θ be the maximum number of mutually non-adjacent arcs in $\mathcal{R}(G)$, and note that $\theta \leq 4$. We analyze the cases $\theta = 4, 3, 2$ in Sections 3,4,5 respectively. When $\theta = 4$ we find that G is $M001$; when $\theta = 3$ we find that G is $M002$; and when $\theta = 2$ we find that G is one of $M001, M002, M417, M428, M437, M462$. We may summarize the results as follows.

Main Theorem. *If G is a maximal exceptional graph in which every vertex has degree less than 28 then G is isomorphic to one of $M001, M002, M417, M428, M437$ and $M462$.*

For the sake of brevity we now introduce the following two arguments related to arcs and curves:

- (the *c-argument*) the presence of an arc $\vec{ij}$ ensures that any edges ik $(k \neq j)$ are red, while any edges jl $(l \neq i)$ are blue;
- (the *d-argument*) the presence of a curve $ijkl$ ensures that any edges pq $(p, q \in \{i, j, k, l\})$ are red, while any edges st $(s, t \notin \{i, j, k, l\})$ are blue.

3. Four arcs

Assumption 1. There exist four mutually non-adjacent arcs in $\mathcal{R}(G)$.

Proposition 1. *If G is a maximal exceptional graph of type (c) and if Assumption 1 holds, then G is the graph $M001$.*

Proof. Without loss of generality, let $\vec{15}$, $\vec{26}$, $\vec{37}$ and $\vec{48}$ be the corresponding arcs.

By the c-argument, the following holds for the edges of $\mathcal{R}(G)$:

(i) edges ij $(1 \le i \le 4,\ 5 \le j \le 8,\ j - i \ne 4)$ do not exist;

(ii) edges ij $(1 \le i < j \le 4)$, if they exist, are red;

(iii) edges ij $(5 \le i < j \le 8)$, if they exist, are blue;

(iv) edges ij $(1 \le i \le 4,\ j = i + 4)$, if they exist, do not have (so far) a prescribed colour.

Making use of the compatibility conditions for arcs we obtain:

(v) arcs $\vec{ij}$ $(1 \le i, j \le 4;\ 5 \le i, j \le 8; 5 \le i \le 8,\ 1 \le j \le 4)$ do not exist,

and consequently

(vi) all arcs which exist are of the form $\vec{ij}$ $(1 \le i \le 4,\ 5 \le j \le 8)$.

From the compatibility conditions for curves (and the maximality of G) we first obtain:

(vii) the curve 1234 does exist;

(viii) for any other curve which exists, whenever it passes through the tail of some arc, it also passes through the head of the arc.

For (vii), note that the curve 1234 is not incompatible with edges (see (ii)-(iv)), arcs (see (vi)), or curves (since by (viii) it has at least two points in common with any other curve).

Recall now that $\deg(\mathbf{e}) \ge \deg(\mathbf{d}_{1234})$. Therefore, there are no more arcs and no more curves adjacent to 1234 (i.e. curves having three points in common with 1234). On the contrary, we have that all edges from (iv) do exist. Moreover, they are either all red or all blue; for if not, there is a curve passing through a red edge, but avoiding a blue one, which is adjacent to the curve 1234 - a contradiction noted above. Consequently, there are no more curves (by the d-argument).

Finally, since there are no more arcs or curves we can easily check that all possible red and/or blue edges from (ii)-(iii) do exist. Thus we arrive at two maximal graphs, each of which we can verify is isomorphic to $M001$ □

4. Three arcs

Assumption 2. The largest set of mutually non-adjacent arcs in $\mathcal{R}(G)$ is of cardinality three.

Proposition 2. *If G is a maximal exceptional graph of type (c) and if Assumption 2 holds, then G is the graph $M002$.*

Proof. Without loss of generality, let $\vec{14}, \vec{25}$ and $\vec{36}$ be the corresponding arcs. By the c-argument we have:

(i) edges ij $(1 \le i \le 3,\ 4 \le j \le 6,\ j - i \ne 3)$ do not exist;

(ii) edges ij $(1 \le i < j \le 3;\ 1 \le i \le 3, 7 \le j \le 8)$, if they exist, are red;

(iii) edges ij $(4 \le i < j \le 6;\ 4 \le i \le 6, 7 \le j \le 8)$, if they exist, are blue;

(iv) edges 14, 25, 36 and 78, if they exist, do not have (so far) a prescribed colour.

Making use of the compatibility conditions for arcs we obtain:

(v) arcs $\vec{ij}$ ($1 \le i,j \le 3$; $4 \le i,j \le 6$; $7 \le i,j \le 8$; $4 \le i \le 6, 1 \le j \le 3 \cup 7 \le j \le 8$; $7 \le i \le 8, 1 \le j \le 3$) do not exist,

and consequently

(vi) all arcs which exist are of the form $\vec{ij}$ ($1 \le i \le 3, 4 \le j \le 8$; $7 \le i \le 8, 4 \le j \le 6$).

From the compatibility conditions for curves (and the maximality of G) we obtain:

(vii) curves 1237 and 1238 do exist;

(viii) for any other curve which exists, whenever it passes through the tail of some arc, it also passes through the head of the arc.

For (vii), note that the curves 1237 and 1238 are not incompatible with edges (see (ii)-(iv)), or arcs (see (vi)), or curves (since by (viii) each these two curves has at least two points in common with any other curve).

We now prove that the (red) edges 12, 13 and 23, and also the (blue) edges 45, 46 and 56 do exist. To this end consider, for example, the pair of edges 12, 45. By (vi), they are not incompatible with arcs. By (viii), 12 can be incompatible only with the curve 3678, and 45 only with the curve 1245. Since the curves 1234, 5678 are incompatible, at least one of the edges exists (by the maximality of G). But then we have $\deg(\mathbf{a}_{12}) > \deg(\mathbf{e})$ (if 12 and 1245 exist), or $\deg(\mathbf{b}_{45}) > \deg(\mathbf{e})$ (if 45 and 3678 exist). In either case we have a contradiction, and so the claim is proved.

Consider now, for example, the sum $\deg(\mathbf{a}_{12}) + \deg(\mathbf{b}_{45})$. By a simple counting argument we have $2\deg(\mathbf{e}) \le \deg(\mathbf{a}_{12}) + \deg(\mathbf{b}_{45})$, with equality if and only if: all edges from (ii)-(iv) do exist; there are no arcs incident with points 1, 2, 4 and 5 other than $\vec{14}, \vec{25}$ and $\vec{36}$; and there are no curves passing through 1 and 2, or avoiding 4 and 5, other than those from (vii). In this case we have $\deg(\mathbf{a}_{12}) = \deg(\mathbf{b}_{45}) = \deg(\mathbf{e})$, and consequently we also know: all edges except the six from (i) do exist; the only arcs are $\vec{14}, \vec{25}$ and $\vec{36}$; there are no more curves except those from (vii) (since at least two of the points $1, 2, 3$, which in turn induce a red triangle, cannot belong to any of them). Finally, we see that the edges 14, 25, 36 and 78 have the same colour, for otherwise (since two edges out of 14, 25 and 36 are coloured by the same colour, say 14 and 25) we find that $\deg(\mathbf{a}_{12}) > \deg(\mathbf{e})$. Thus we arrive at two maximal graphs, each of which we can verify is isomorphic to $M002$. □

5. Two arcs

Assumption 3. The largest set of mutually non-adjacent arcs in $\mathcal{R}(G)$ is of cardinality two.

Proposition 3. *If G is a maximal exceptional graph of type (c), and if Assumption* 3 *holds, then G is one of the following graphs: $M001$, $M002$, $M417$, $M428$, $M437$ and $M462$.*

Proof. Without loss of generality, let $\vec{13}$ and $\vec{24}$ be the corresponding arcs. By the c-argument we have:

(i) edges 14 and 23 do not exist;

(ii) edges 12 and ij $(1 \leq i \leq 2,\ 5 \leq j \leq 8)$, if they exist, are red;

(iii) edges 34 and ij $(3 \leq i \leq 4,\ 5 \leq j \leq 8)$, if they exist, are blue;

(iv) edges 13, 24 and ij $(5 \leq i, j \leq 8)$, if they exist, do not have (so far) any prescribed colour.

Making use of the compatibility conditions for arcs (and Assumption 3 as well) we obtain:

(v) arcs $\vec{12}$, $\vec{21}$, $\vec{34}$, $\vec{43}$, $\vec{31}$, $\vec{32}$, $\vec{41}$, $\vec{42}$ and $\vec{ij}$ $(5 \leq i, j \leq 8;\ 5 \leq i \leq 8,\ 1 \leq j \leq 2;\ 3 \leq i \leq 4,\ 5 \leq j \leq 8)$ do not exist,

and consequently,

(vi) all arcs which exist are of the form $\vec{ij}$ $(1 \leq i \leq 2; 3 \leq j \leq 8)$, $(5 \leq i \leq 8; 3 \leq j \leq 4)$.

Note also that:

(vii) any curve passing through the tail of an arc also passes through the head of the same arc.

Claim 1: *The edges* 12 *and* 34 *do exist.*

Suppose that the (red) edge 12 does not exist. Since it is compatible with all arcs (see (vi)), there is a curve incompatible with it. But such a curve avoids the points 1 and 2, and hence (by (vii)) also the points 3 and 4; accordingly it must be the curve 5678. Next suppose that the (blue) edge 34 does not exist. Since it is compatible with all arcs (see (vi)), there is a curve incompatible with it. But such a curve passes through points 3 and 4, and hence (by (vii)) also the points 1 and 2; accordingly it must be the curve 1234. Since the curves 1234, 5678 are incompatible, at least one of the edges 12 and 34 does exist. If only one exists then we obtain a contradiction: either $\deg(\mathbf{a}_{12}) > \deg(\mathbf{e})$ (if 12 exists), or $\deg(\mathbf{b}_{34}) > \deg(\mathbf{e})$ (if 34 exists). The claim follows.

Let us now introduce some more notation: A' (A'') is the number of arcs incident with one (resp. two) of the points $1, 2, 3, 4$, while C' (C'') is the number of curves adjacent to one (resp. both) of the edges $12,\ 34$; E is the number of edges from (iv).

Claim 2: *with the above notation the following holds:*

$$A' + 2A'' + C' + 2C'' \leq E. \tag{1}$$

Since $2\deg(\mathbf{e}) \geq \deg(\mathbf{a}_{12}) + \deg(\mathbf{b}_{45})$, (1) follows from a simple counting argument.

We now turn our attention to the subgraph (say H) induced by vertices $5, 6, 7, 8$ (notice that it contains only coloured edges). For the sake of brevity, we refer to red edges as edges of R-type, to blue edges as edges of B-type, and to non-edges as edges of N-type. Two edges of H will be called *opposite* if they are disjoint.

Claim 3: *At most one of the arcs* $\vec{14}$ *and* $\vec{23}$ *exists.*

If both arcs $\vec{14}$ and $\vec{23}$ exist, then (by the c-argument) the edges 13 and 24 do not exist, and so $E \leq 6$. On the other hand, $A'' \geq 4$, and consequently $E = 8$, a contradiction.

Claim 4: *If H contains a pair of opposite edges of different types, then there exists a curve $12ij$, where ij is of R-type (if any), or of N-type (otherwise).*

By the maximality of G, the curve $12ij$ exists if there is no arc or curve incompatible with it. By (vi) no arc is incompatible. Without loss of generality, assume that $ij = 56$, and suppose that c is a curve incompatible with the curve 1256. If c is disjoint from 1256, then $c = 3478$ and we obtain a contradiction at once (either edge 78 is blue or edge 56 red). Otherwise c has one point in common with 1256. Since 12 is red, 5 and 6 cannot be such points. Accordingly 1 or 2, say 1, should be considered. Since 2 is not on c, the same applies for 4 (note that $\vec{24}$ is an arc). Then we have $c = 1378$, and we obtain the same contradiction as in the previous situation.

Claim 5: *If i and j are non-adjacent in H, then there exist two curves, one passing through i and j, the other avoiding i and j.*

As in the proof of [11, Theorem 3.6], i and j are non-adjacent due to the existence of one the following pairs of objects: (a) two arcs; (b) an arc and a curve; (c) two curves. Accordingly we need to show the impossibility of (a) and (b). Without loss of generality, take $ij = 56$.

To reject (a) consider arcs $\vec{s5}$ and $\vec{6t}$ (where $s \in \{1, 2\}$, $t \in \{3, 4\}$). Then $A' \geq 2$. Since the edges 57 and 58 (if they exist) are red, and edges 67 and 68 (if they exist) are blue, by Claim 4 (note that $E \geq 4$) we have $C'' \geq 1$ Then by (1), $E = 8$, while actually $E \leq 7$.

To reject (b) we may suppose (by considering $\phi(\mathcal{R}(G))$ if necessary) that there exist an arc $\vec{s5}$ and a curve $s56t$ ($s \in \{1, 2\}$ is fixed, while t is chosen appropriately). Then $A' \geq 1$. Since 57 and 58 (if they exist) are blue, and since 67 and 68 are not both blue (otherwise $\deg(\mathbf{a}_{12}) > \deg(\mathbf{e})$), we have $C'' \geq 1$ (by Claim 4 – note that $E \geq 4$). But then, by (1), $E \geq 7$, and so $E = 7$. Consequently, two pairs of opposite edges of different types can be found, as in Claim 4, and hence $C'' \geq 2$, a contradiction.

Claim 6: *If one of two opposite edges in H is of N-type, then both are of N-type.*

Without loss of generality, let points 5 and 6 be non-adjacent. Then, by Claim 4, there exists a curve 1256. By Claim 5, there is a curve avoiding points 5 and 6. To be compatible with the former curve, it passes through points 1 and 2. But then $C'' \geq 2$, and hence $E = 8$, a contradiction (since $E < 8$).

In what follows we shall consider several cases depending on the (coloured) subgraph H.

Case 1: *There exist (in H) two opposite edges of N-type.*

Without loss of generality, let 56 and 78 be edges of N-type, i.e. non-edges. Since 5 and 6 are non-adjacent, there exists (by Claim 5) a curve c_1 passing through them (and a curve c_1' avoiding them). In addition, c_1 passes through at least one of the points 1 and 2 (since 12 is red), but not both. (Otherwise, c_1 and c_1' both pass through points 1 and 2, whence $C'' \geq 2$ and $E = 8$, a contradiction since $E \leq 6$). Similarly, since 7 and 8 are non-adjacent, there exists a curve c_2 passing through them (and a curve c_2' avoiding them). Next, as above, c_2 passes through exactly one of the points $1, 2$. If c_1 and c_2 do not pass through the same point of the edge 12, then they share at most one common point, a contradiction. Accordingly assume that both curves, c_1 and c_2, pass through the point 1. Thus $c_1 = 156x$ and $c_2 = 178y$ ($x, y \neq 2$). Next, $x, y \neq 4$ because $\vec{24}$ is an arc. By the compatibility of c_1 and c_2, $x = 3$ if and only if $y = 3$. Consequently $x \in \{7, 8\}$ if and only if $y \in \{5, 6\}$. The latter possibility is ruled out, for otherwise $\deg(\mathbf{b}_{34}) > \deg(\mathbf{e})$ (notice that an edge 24 (if exists) is blue, while with at most one exception, the edges of H are red). Accordingly we consider only the former possibility: then $c_1 = 1356$, while $c_2 = 1378$.

We next consider the arc $\vec{14}$ (note that the arc $\vec{23}$ is excluded by c_1 or c_2). The arc $\vec{14}$ can be excluded only by a curve passing through 4 (and hence 2) and avoiding $1, 3$. But this curve is incompatible with c_1 or c_2, and so the arc $\vec{14}$ exists (by maximality). Then $A'' = 3$, and by (1), $E \geq 6$. Then $E = 6$, $A' = 0$ and $C' = C'' = 0$. Thus all edges from (iv) except 56 and 78 do exist. Then, clearly, 13 is red, while 24 is blue. By Claim 4, if two opposite edges of H are of different types then $C' > 0$ or $C'' > 0$ – a contradiction to the above. If all four edges of H are coloured by the same colour, then either $\deg(\mathbf{a}_{12}) > \deg(\mathbf{e})$ (all are blue) or $\deg(\mathbf{b}_{34}) > \deg(\mathbf{e})$ (all are red). Finally, there are no more arcs (see (vi) and recall that $A' = 0$), and no more curves (each one if added gives $C' > 0$ or $C'' > 0$). By the d-argument (see (ii) and (iii)), the edges ij $(i \in \{2, 4\},\ j \in \{5, 6, 7, 8\})$ do not exist. Thus we arrive at a 22-vertex graph which we can verify is isomorphic to $M001$.

In view of Claim 6 we may now suppose that H is a complete graph (on four vertices), i.e. there are no edges of N-type. We distinguish cases according to the number $n_b(H)$ of blue edges, which we may assume (considering $\phi(\mathcal{R}(G))$ if necessary) is not less than the number $n_r(H)$ of red edges.

Case 2: $n_b(H) = 6$, $n_r(H) = 0$.

We first observe that the edges 13 and 24 are blue (for otherwise $\deg(\mathbf{a}_{12}) > \deg(\mathbf{e})$). Next we find at once that there are no more arcs because no point can be a head. Also there are no curves, since each curve can include at most one point common with H (due to its colouring) and at most two of the remaining points (again due to the colouring pattern). On the other hand, all red and blue edges from (ii) and (iii) do exist. Thus we obtain a 29-vertex graph which we can verify is isomorphic to $M428$.

Case 3: $n_b(H) = 5$, $n_r(H) = 1$.

By Claim 4, we know that a curve 1256 exists. It follows that the edges 13 and 24 are blue (for otherwise $\deg(\mathbf{a}_{12}) > \deg(\mathbf{e})$). Next, there are no more arcs, and no curves, by arguments are similar to those in Case 2. On the other hand, all red and blue edges from (ii) and (iii) do exist, and we obtain a 30-vertex graph which we can verify is isomorphic to $M437$.

Case 4: $n_b(H) = 4$, $n_r(H) = 2$.

Here we distinguish two subcases depending on the factorization of H induced by the edge-colouring.

Subcase $4a$**:** *Two red edges are non-adjacent.*

We now assume, without loss of generality, that edges 57 and 68 are red (so that the edges 56, 67, 78 and 85 are blue). Consider first the arcs. By (vi), the only further possible arcs are $\vec{14}$, $\vec{23}$; and by Claim 3, at most one of them is present. On the other hand, there are no curves at all (due to the colouring of H).

Assume first that an arc, say $\vec{14}$, exists. By the maximality of G, both of the edges 13, 24 exist, and clearly the former is red, while the latter blue; in addition, all edges from (ii) and (iii) exist. Thus we obtain a 30-vertex graph which we can verify is isomorphic to $M437$.

Assume now that neither of the arcs $\vec{14}, \vec{23}$ exists (so that no more arcs are present). Then the edges 13, 24 are either both blue, or both red. By the maximality of G, it follows that all edges from (ii) and (iii) do exist. Thus we obtain (in the "blue case") a graph isomorphic to $M417$, and (in the "red case") a graph isomorphic to $M428$.

Subcase $4b$**:** *Two red edges are adjacent.*

We now assume, without loss of generality, that the edges 56 and 67 are red (so that the edges 57, 58, 68 and 78 are blue). By Claim 4, since the edges 58 and 78 are blue, we know that the curves 1256 and 1267 exist. Therefore, by (1), $E = 8$ and in particular the edges 13, 24 exist. Moreover, they are both blue (for otherwise, $\deg(\mathbf{a}_{12}) > \deg(\mathbf{e})$). Next, by (1), there are no more arcs (see (vi)): and as in Case 2 there are no more curves. On the other hand, all edges from (ii) and (iii) do exist. Thus we obtain a 31-vertex graph which we can verify is isomorphic to $M462$.

Case 5: $n_b(H) = 3$, $n_r(H) = 3$.

Again we distinguish two subcases depending on the factorization of H induced by the edge-colouring.

Subcase 5*a*: *Three red (or three blue) edges form a path.*

Without loss of generality, assume that edges 56, 58 and 67 are red (forming a path). The only further possible arcs are $\vec{14}$, $\vec{23}$ (cf. (vi)), and by Claim 3 at most one of these is present. By Claim 4, the curve 1256 does exist.

Assume first that an arc, say $\vec{14}$, exists. Then $A'' \geq 3$, while $C'' \geq 1$, and hence $E = 8$. Then the edges 13 and 24 exist; the former is red and the latter is blue, by the c-argument. We next consider curves. Clearly, each curve must pass through points 5 and 6 (due to the colouring of H); and each curve must pass through 1, for otherwise, since 12 and 13 are red, the curve passes through 2 and 3 and is therefore incompatible with $\vec{13}$. For the fourth point we must take 3 to obtain a new curve. Now we have two possibilities: the curve 1356 exists (and consequently the edges 27, 28, 35 and 36 do not exist), or the curve 1356 does not exist (while the edges 27, 28, 35 and 36 do exist). In both cases all other edges from (ii) and (iii) do exist. In the former case we obtain a graph isomorphic to $M002$, and in the latter case a graph isomorphic to $M462$.

Assume next that neither of the arcs $\vec{14}$ and $\vec{23}$ exists (so that no more arcs are present). Then, by arguments similar to those above, the following two curves are possible: 1356 and 2456. We consider the possibilities in turn.

If both curves 1356 and 2456 exist, then the edges 13, 24, 17, 18, 27, 28, 35, 36, 45, 46 do not exist. Next, all other edges from (ii) and (iii) do exist, and we obtain a graph isomorphic to $M001$.

If only one of the curves 1356 and 2456 exists, say 1356, then an arc $\vec{14}$ can be added (13, if it exists, is red, while 24, if it exists, is blue), and hence we obtain a contradiction to to current assumptions.

If neither of the curves 1356 and 2456 exists, then the edges 13 and 24 are either both blue, or both red. Considering $\phi(\mathcal{R}(G))$ if necessary, we may assume that the former holds. Next, all edges from (ii) and (iii) do exist, and so we obtain a 30-vertex graph which we can verify is isomorphic to $M437$.

Subcase 5*b*: *Three red (or blue) edges form a star.*

Without loss of generality, assume that edges 56, 57 and 58 are red edges (forming a star). Next, we have that $A'' \geq 2$ and $C'' \geq 2$ (since curves 1256, 1257 and 1258 exist, by Claim 4). But then, by Claim 2, $E > 8$, a contradiction. □

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